RIGIDITY OF MANIFOLDS WITH BOUNDARY UNDER A LOWER BAKRY-ÉMERY RICCI CURVATURE BOUND

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Abstract. We study Riemannian manifolds with boundary under a lower Bakry-Émery Ricci curvature bound. In our weighted setting, we prove several rigidity theorems for such manifolds with boundary. We conclude a rigidity theorem for the inscribed radii, a volume growth rigidity theorem for the metric neighborhoods of the boundaries, and various splitting theorems. We also obtain rigidity theorems for the smallest Dirichlet eigenvalues for the weighted p-Laplacians.

1. Introduction

For Riemannian manifolds without boundary, under a lower Bakry-Émery Ricci curvature bound, we know several comparison results and rigidity theorems (see e.g., [12], [32], [38] and [45]). For metric measure spaces, Lott and Villani [33], [34], and Sturm [42], [43] have introduced the so-called curvature dimension condition that is equivalent to a lower Bakry-Émery Ricci curvature bound for manifolds without boundary. Under a curvature dimension condition, they have obtained comparison results in [34] and [42]. Under a more restricted condition, Gigli [15], and Ketterer [24], [25] have recently studied rigidity theorems.

In this paper, we study Riemannian manifolds with boundary under a lower Bakry-Émery Ricci curvature bound, and under a lower mean curvature bound for the boundary. For such manifolds with boundary, we obtain several comparison results, and we prove rigidity theorems. In an unweighted standard setting, for instance, Heintze and Karcher [17], and Kasue [19] have obtained comparison results, and Kasue [20], [21], and the author [40] have done rigidity theorems. We generalize them in our weighted setting.

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1.1. **Setting.** For $n \geq 2$, let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g. The boundary ∂M is assumed to be smooth. We denote by d_M the Riemannian distance on M induced from the length structure determined by g. Let $f : M \to \mathbb{R}$ be a smooth function. For the Riemannian volume measure vol_g on M induced from g, we put $m_f := e^{-f} \mathrm{vol}_g$.

We denote by Ric_g the Ricci curvature on M defined by g. We denote by ∇f the *gradient* of f, and by $\mathrm{Hess}\, f$ the *Hessian* of f. For $N \in (-\infty, \infty]$, the *Bakry-Émery Ricci curvature* Ric_f^N is defined as follows ([2], [38]): If $N \in (-\infty, \infty) \setminus \{n\}$, then

$$\mathrm{Ric}_f^N := \mathrm{Ric}_g + \mathrm{Hess}\, f - \frac{\nabla f \otimes \nabla f}{N-n};$$

if $N = \infty$, then $\mathrm{Ric}_f^N := \mathrm{Ric}_g + \mathrm{Hess}\, f$; if $N = n$, and if f is a constant function, then $\mathrm{Ric}_f^N := \mathrm{Ric}_g$; if $N = n$, and if f is not constant, then put $\mathrm{Ric}_f^N := -\infty$. For $K \in \mathbb{R}$, by $\mathrm{Ric}_{f,M}^N \geq K$ we mean that the infimum of Ric_f^N on the unit tangent bundle on the interior $\mathrm{Int}\, M$ of M is at least K. For $x \in \partial M$, we denote by H_x the mean curvature on ∂M at x in M defined as the trace of the shape operator for the unit inner normal vector u_x at x. The *f-mean curvature* $H_{f,x}$ at x is defined by

$$H_{f,x} := H_x + g((\nabla f)_x, u_x).$$

For $\Lambda \in \mathbb{R}$, by $H_{f,\partial M} \geq \Lambda$ we mean $\inf_{x \in \partial M} H_{f,x} \geq \Lambda$. The subject of our study is a metric measure space (M, d_M, m_f) such that for $N \in [n, \infty)$, and for $\kappa, \lambda \in \mathbb{R}$, we have $\mathrm{Ric}_{f,M}^N \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$, or such that $\mathrm{Ric}_{f,M}^\infty \geq 0$ and $H_{f,\partial M} \geq 0$.

1.2. **Inscribed radius rigidity.** For $\kappa \in \mathbb{R}$, we denote by M_κ^n the n-dimensional space form with constant curvature κ. We say that $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the *ball-condition* if there exists a closed geodesic ball $B_{\kappa,\lambda}^n$ in M_κ^n with non-empty boundary $\partial B_{\kappa,\lambda}^n$ such that $\partial B_{\kappa,\lambda}^n$ has a constant mean curvature $(n-1)\lambda$. We denote by $C_{\kappa,\lambda}$ the radius of $B_{\kappa,\lambda}^n$. We see that κ and λ satisfy the ball-condition if and only if either (1) $\kappa > 0$; (2) $\kappa = 0$ and $\lambda > 0$; or (3) $\kappa < 0$ and $\lambda > \sqrt{|\kappa|}$. Let $s_{\kappa,\lambda}(t)$ be a unique solution of the so-called Jacobi-equation

$$\phi''(t) + \kappa \phi(t) = 0$$

with initial conditions $\phi(0) = 1$ and $\phi'(0) = -\lambda$. We see that κ and λ satisfy the ball-condition if and only if the equation $s_{\kappa,\lambda}(t) = 0$ has a positive solution; in particular, $C_{\kappa,\lambda} = \inf\{t > 0 \mid s_{\kappa,\lambda}(t) = 0\}$.

Let $\rho_{\partial M} : M \to \mathbb{R}$ be the distance function from ∂M defined as $\rho_{\partial M}(p) := d_M(p, \partial M)$. The *inscribed radius of* M is defined as

$$D(M, \partial M) := \sup_{p \in M} \rho_{\partial M}(p).$$

We have the following rigidity theorem for the inscribed radius:

Theorem 1.1. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let f be a smooth function on M. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then we have $D(M, \partial M) \leq C_{\kappa,\lambda}$. Moreover, if there exists $p \in M$ such that $\rho_{\partial M}(p) = C_{\kappa,\lambda}$, then (M, d_M) is isometric to $(B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})$ and $N = n$; in particular, f is constant on M.*

Kasue [20] has proved Theorem 1.1 in the standard case where $f = 0$ and $N = n$. We prove Theorem 1.1 in a similar way to that in [20].

Remark 1.1. M. Li [28] later than [20] has proved Theorem 1.1 when $f = 0$, $N = n$ and $\kappa = 0$. H. Li and Wei have proved Theorem 1.1 in [27] when $\kappa = 0$, and in [26] when $\kappa < 0$. In [26] and [27], Theorem 1.1 in the specific cases have been proved in a similar way to that in [28].

1.3. **Volume growth rigidity.** For $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, if κ and λ satisfy the ball-condition, then we put $\bar{C}_{\kappa,\lambda} := C_{\kappa,\lambda}$; otherwise, $\bar{C}_{\kappa,\lambda} := \infty$. We define a function $\bar{s}_{\kappa,\lambda} : [0, \infty) \to \mathbb{R}$ by

$$\bar{s}_{\kappa,\lambda}(t) := \begin{cases} s_{\kappa,\lambda}(t) & \text{if } t < \bar{C}_{\kappa,\lambda}, \\ 0 & \text{if } t \geq \bar{C}_{\kappa,\lambda}. \end{cases}$$

For $N \in [2, \infty)$, we define a function $s_{N,\kappa,\lambda} : (0, \infty) \to \mathbb{R}$ by

$$s_{N,\kappa,\lambda}(r) := \int_0^r \bar{s}^{N-1}_{\kappa,\lambda}(t)\, dt.$$

For $r > 0$, we put $B_r(\partial M) := \{\, p \in M \mid \rho_{\partial M}(p) \leq r \,\}$. For $x \in \partial M$, let $\gamma_x : [0, T) \to M$ be the geodesic with initial conditions $\gamma_x(0) = x$ and $\gamma'_x(0) = u_x$. We denote by h the induced Riemnnian metric on ∂M. For the Riemannian volume measure vol_h on ∂M induced from h, we put $m_{f,\partial M} := e^{-f|_{\partial M}} \mathrm{vol}_h$. For an interval I, and for a connected component ∂M_1 of ∂M, let $I \times_{\kappa,\lambda} \partial M_1$ denote the warped product $(I \times \partial M_1, dt^2 + s^2_{\kappa,\lambda}(t) h)$. We put $I_{\kappa,\lambda} := [0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, and denote by $d_{\kappa,\lambda}$ the Riemannian distance on $I_{\kappa,\lambda} \times_{\kappa,\lambda} \partial M$.

We obtain relative volume comparison theorems of Bishop-Gromov type for the metric neighborhoods of the boundaries (see Theorems 5.4 and 5.5). We conclude rigidity theorems concerning the equality cases in those comparison theorems (see Subsection 5.3).

One of the volume growth rigidity results is the following:

Theorem 1.2. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. For $N \in [n,\infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. If we have*

$$\liminf_{r\to\infty} \frac{m_f(B_r(\partial M))}{s_{N,\kappa,\lambda}(r)} \geq m_{f,\partial M}(\partial M), \tag{1.1}$$

then (M, d_M) is isometric to $(I_{\kappa,\lambda} \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$, and for every $x \in \partial M$ we have $f \circ \gamma_x = f(x) - (N-n) \log s_{\kappa,\lambda}$ on $I_{\kappa,\lambda}$. Moreover, if κ and λ satisfy the ball-condition, then (M, d_M) is isometric to $(B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})$ and $N = n$; in particular, f is constant on M.

In [40], Theorem 1.2 has been proved when $f = 0$ and $N = n$.
In the case of $N = \infty$, we have the following:

Theorem 1.3. *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Suppose $\mathrm{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. If we have*

$$\liminf_{r\to\infty} \frac{m_f(B_r(\partial M))}{r} \geq m_{f,\partial M}(\partial M), \tag{1.2}$$

then (M, d_M) is isometric to $([0,\infty) \times \partial M, d_{[0,\infty)\times\partial M})$.

Remark 1.2. On one hand, under the same setting as in Theorem 1.2, we always have the following (see Lemma 5.2):

$$\limsup_{r\to\infty} \frac{m_f(B_r(\partial M))}{s_{N,\kappa,\lambda}(r)} \leq m_{f,\partial M}(\partial M). \tag{1.3}$$

On the other hand, under the same setting as in Theorem 1.3, we always have the following (see Lemma 5.3):

$$\limsup_{r\to\infty} \frac{m_f(B_r(\partial M))}{r} \leq m_{f,\partial M}(\partial M). \tag{1.4}$$

Theorems 1.2 and 1.3 are concerned with rigidity phenomena.

Remark 1.3. In the forthcoming paper [41], we prove the same result as Theorem 1.3 under a weaker assumption that $\mathrm{Ric}^N_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$ for $N < 1$. In the rigidity case, we prove further that for every $x \in \partial M$ the function $f \circ \gamma_x$ is constant on $[0,\infty)$ (see Theorem 1.1 in [41]).

1.4. **Splitting theorems.** Define a function $\tau : \partial M \to \mathbb{R} \cup \{\infty\}$ by

$$\tau(x) := \sup\{\, t \in (0, \infty) \mid \rho_{\partial M}(\gamma_x(t)) = t \,\}. \tag{1.5}$$

We obtain the following splitting theorem:

Theorem 1.4. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in [n, \infty)$, we suppose $\operatorname{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. If for some $x_0 \in \partial M$ we have $\tau(x_0) = \infty$, then (M, d_M) is isometric to $([0, \infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$, and for all $x \in \partial M$ and $t \in [0, \infty)$ we have $(f \circ \gamma_x)(t) = f(x) + (N - n)\lambda t$.*

In the standard case where $f = 0$ and $N = n$, Kasue [20] has proved Theorem 1.4 under the assumption that the boundary is compact (see also the work of Croke and Kleiner [11]). In the standard case, Theorem 1.4 itself has been proved in [40].

In the case of $N = \infty$, we have the following splitting theorem:

Theorem 1.5. *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function such that $\sup f(M) < \infty$. Suppose $\operatorname{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. If for some $x_0 \in \partial M$ we have $\tau(x_0) = \infty$, then the metric space (M, d_M) is isometric to $([0, \infty) \times \partial M, d_{[0,\infty)\times\partial M})$.*

Remark 1.4. In Theorem 1.5, we need the assumption $\sup f(M) < \infty$. We denote by $\mathbb{S}^{n-1}$ the $(n-1)$-dimensional standard unit sphere, and by ds^2_{n-1} the canonical metric on $\mathbb{S}^{n-1}$. We put

$$M := \left([0, \infty) \times \mathbb{S}^{n-1}, dt^2 + \cosh^2 t \, ds^2_{n-1}\right).$$

Let f be a function on M defined by $f(p) := (n-1)\rho_{\partial M}(p)^2$. Then for all $x \in \partial M$ we have $H_{f,x} = H_x = 0$. Take $p \in \operatorname{Int} M$, and put $l := \rho_{\partial M}(p)$. We choose an orthonormal basis of $\{e_i\}_{i=1}^n$ of T_pM such that $e_n = \nabla \rho_{\partial M}$. For all $i = 1, \dots, n-1$, we have

$$\operatorname{Ric}_g(e_i, e_i) = (n-2)\frac{1 - \sinh^2 l}{\cosh^2 l} - 1, \ \operatorname{Hess} f(e_i, e_i) = 2(n-1)l\frac{\sinh l}{\cosh l},$$

and $\operatorname{Ric}_g(e_n, e_n) = -(n-1)$, $\operatorname{Hess} f(e_n, e_n) = 2(n-1)$. For all $i, j = 1, \dots, n$ with $i \neq j$, we have $\operatorname{Ric}_g(e_i, e_j) = 0$ and $\operatorname{Hess} f(e_i, e_j) = 0$. From direct computations, it follows that if $n \geq 3$, then $\operatorname{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. On the other hand, M is not isometric to the direct product $[0, \infty) \times \mathbb{S}^{n-1}$.

Remark 1.5. In [41], we prove the same splitting theorem as Theorem 1.5 under a weaker assumption that $\operatorname{Ric}^N_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$ for

$N < 1$. In the splitting case, we show that for every $x \in \partial M$ the function $f \circ \gamma_x$ is constant (see Theorem 1.3 in [41]).

In Theorems 1.4 and 1.5, by applying the splitting theorems of Cheeger-Gromoll type (cf. [10]) to the boundary, we obtain the multi-splitting theorems (see Subsection 6.3). We also generalize the splitting theorems studied in [20] (and [11], [18]) for manifolds with boundary whose boundaries are disconnected (see Subsection 6.4).

1.5. **Eigenvalue rigidity.** For $p \in [1,\infty)$, the $(1,p)$-*Sobolev space* $W^{1,p}_0(M,m_f)$ *on* (M,m_f) *with compact support* is defined as the completion of the set of all smooth functions on M whose support is compact and contained in $\operatorname{Int} M$ with respect to the standard $(1,p)$-Sobolev norm. We denote by $\|\cdot\|$ the standard norm induced from g, and by div the divergence with respect to g. For $p \in [1,\infty)$, the (f,p)-*Laplacian* $\Delta_{f,p}\,\phi$ for $\phi \in W^{1,p}_0(M,m_f)$ is defined by

$$\Delta_{f,p}\,\phi := -e^{f} \operatorname{div} \left(e^{-f} \|\nabla \phi\|^{p-2}\, \nabla \phi\right)$$

as a distribution on $W^{1,p}_0(M,m_f)$. A real number μ is said to be an (f,p)-*Dirichlet eigenvalue* for $\Delta_{f,p}$ on M if there exists a non-zero function $\phi \in W^{1,p}_0(M,m_f)$ such that $\Delta_{f,p}\phi = \mu |\phi|^{p-2}\,\phi$ holds on $\operatorname{Int} M$ in a distribution sense on $W^{1,p}_0(M,m_f)$. For $p \in [1,\infty)$, the *Rayleigh quotient* $R_{f,p}(\phi)$ for $\phi \in W^{1,p}_0(M,m_f) \setminus \{0\}$ is defined as

$$R_{f,p}(\phi) := \frac{\int_M \|\nabla \phi\|^p \, d\, m_f}{\int_M |\phi|^p \, d\, m_f}.$$

We put $\mu_{f,1,p}(M) := \inf_{\phi} R_{f,p}(\phi)$, where the infimum is taken over all non-zero functions in $W^{1,p}_0(M,m_f)$. The value $\mu_{f,1,2}(M)$ is equal to the infimum of the spectrum of $\Delta_{f,2}$ on (M,m_f). If M is compact, and if $p \in (1,\infty)$, then $\mu_{f,1,p}(M)$ is equal to the infimum of the set of all (f,p)-Dirichlet eigenvalues on M.

Let $p \in (1,\infty)$. For $N \in [2,\infty)$, $\kappa,\lambda \in \mathbb{R}$, and $D \in (0,\bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, let $\mu_{p,N,\kappa,\lambda,D}$ be the positive minimum real number μ such that there exists a non-zero function $\phi : [0,D] \to \mathbb{R}$ satisfying

$$\begin{aligned}(1.6)\quad \left(|\phi'(t)|^{p-2}\phi'(t)\right)' + (N-1)\frac{s'_{\kappa,\lambda}(t)}{s_{\kappa,\lambda}(t)}(|\phi'(t)|^{p-2}\phi'(t))& \\ + \mu\,|\phi(t)|^{p-2}\phi(t) = 0, \quad \phi(0) = 0, \quad \phi'(D) = 0.&\end{aligned}$$

For $D \in (0,\infty)$, let $\mu_{p,\infty,D}$ be the positive minimum real number μ such that there exists a non-zero function $\phi : [0,D] \to \mathbb{R}$ satisfying

$$(1.7)\quad \left(|\phi'(t)|^{p-2}\phi'(t)\right)' + \mu\,|\phi(t)|^{p-2}\phi(t) = 0, \quad \phi(0) = 0, \quad \phi'(D) = 0.$$

We recall the notion of model spaces that has been introduced by Kasue in [21] in our setting. We say that $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the *model-condition* if the equation $s'_{\kappa,\lambda}(t) = 0$ has a positive solution. We see that κ and λ satisfy the model-condition if and only if either (1) $\kappa > 0$ and $\lambda < 0$; (2) $\kappa = 0$ and $\lambda = 0$; or (3) $\kappa < 0$ and $\lambda \in (0, \sqrt{|\kappa|})$.

Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition or the model-condition. Suppose that M is compact. For κ and λ satisfying the model-condition, we define a positive number $D_{\kappa,\lambda}(M)$ as follows: If $\kappa = 0$ and $\lambda = 0$, then $D_{\kappa,\lambda}(M) := D(M, \partial M)$; otherwise, $D_{\kappa,\lambda}(M) := \{t > 0 \mid s'_{\kappa,\lambda}(t) = 0\}$. We say that (M, d_M) is a (κ,λ)-*equational model space* if M is isometric to either (1) for κ and λ satisfying the ball-condition, the closed geodesic ball $B^n_{\kappa,\lambda}$; (2) for κ and λ satisfying the model-condition, and for a connected component ∂M_1 of ∂M, the warped product $[0, 2D_{\kappa,\lambda}(M)] \times_{\kappa,\lambda} \partial M_1$; or (3) for κ and λ satisfying the model-condition, and for an involutive isometry σ of ∂M without fixed points, the quotient space $([0, 2D_{\kappa,\lambda}(M)] \times_{\kappa,\lambda} \partial M)/G_\sigma$, where G_σ is the isometry group on $[0, 2D_{\kappa,\lambda}(M)] \times_{\kappa,\lambda} \partial M$ whose elements consist of the identity and the involute isometry $\hat{\sigma}$ defined by $\hat{\sigma}(t, x) := (2D_{\kappa,\lambda}(M) - t, \sigma(x))$.

Let $p \in (1, \infty)$. Let M be a (κ,λ)-equational model space. From a standard argument, we see that if M is isometric to $B^n_{\kappa,\lambda}$, then $\mu_{0,1,p}(M) = \mu_{p,n,\kappa,\lambda,C_{\kappa,\lambda}}$. Furthermore, if M is not isometric to $B^n_{\kappa,\lambda}$, then $\mu_{0,1,p}(M) = \mu_{p,n,\kappa,\lambda,D_{\kappa,\lambda}(M)}$ for the corresponding κ, λ and $D_{\kappa,\lambda}(M)$.

We establish the following rigidity theorem for $\mu_{f,1,p}$:

Theorem 1.6. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let $p \in (1, \infty)$. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, we assume $D(M, \partial M) \leq D$. Then we have*

$$\mu_{f,1,p}(M) \geq \mu_{p,N,\kappa,\lambda,D}. \tag{1.8}$$

If the equality in (1.8) holds, then (M, d_M) is a (κ,λ)-equational model space; more precisely, the following hold:

1. *if $D = \bar{C}_{\kappa,\lambda}$, then κ and λ satisfy the ball-condition, (M, d_M) is isometric to $(B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})$, and $N = n$; in particular, f is constant on M;*
2. *if $D \in (0, \bar{C}_{\kappa,\lambda})$, then κ and λ satisfy the model-condition, (M, d_M) is a (κ,λ)-equational model space, and $f \circ \gamma_x = f(x) - (N-n) \log s_{\kappa,\lambda}$ on $[0, D_{\kappa,\lambda}(M)]$ for all $x \in \partial M$.*

Kasue [21] has proved Theorem 1.6 when $p = 2, f = 0$ and $N = n$. It seems that the method of the proof in [21] does not work in our

non-linear case of $p \neq 2$ (see Remark 7.3). We prove Theorem 1.6 by a global Laplacian comparison result for $\rho_{\partial M}$ (see Proposition 3.7) and an inequality of Picone type for the p-Laplacian (see Lemma 7.1).

In the case of $N = \infty$, we have the following:

Theorem 1.7. *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let $p \in (1, \infty)$. Suppose $\mathrm{Ric}^{\infty}_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. For $D \in (0, \infty)$, we assume $D(M, \partial M) \leq D$. Then we have*

$$\mu_{f,1,p}(M) \geq \mu_{p,\infty,D}. \tag{1.9}$$

If the equality in (1.9) *holds, then the metric space (M, d_M) is a $(0,0)$-equational model space, and $D(M, \partial M) = D$.*

Remark 1.6. In [41], we prove the same rigidity result as Theorem 1.7 under a weaker assumption that $\mathrm{Ric}^{N}_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$ for $N < 1$. In the rigidity case, we also prove that for every $x \in \partial M$ the function $f \circ \gamma_x$ is constant on $[0, D]$ (see Theorem 1.5 in [41]).

In Theorems 1.6 and 1.7, we have explicit lower bounds for $\mu_{f,1,p}$ (see Subsection 7.3).

We show some volume estimates for a relatively compact domain in M (see Propositions 8.1 and 8.2). From the volume estimates, we derive lower bounds for $\mu_{f,1,p}$ for manifolds with boundary that are not necessarily compact (see Theorems 8.4 and 8.5). By using the estimate for $\mu_{f,1,p}$, and by using Theorem 1.4, we obtain the following:

Theorem 1.8. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary. Let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $p \in (1, \infty)$. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^{N}_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then we have*

$$\mu_{f,1,p}(M) \geq \left(\frac{(N-1)\lambda}{p} \right)^{p}. \tag{1.10}$$

If the equality in (1.10) *holds, then the metric space (M, d_M) is isometric to $([0, \infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$, and for all $x \in \partial M$ and $t \in [0, \infty)$ we have $(f \circ \gamma_x)(t) = f(x) + (N-n)\lambda t$.*

Theorem 1.8 has been proved in [40] in the standard case where $f = 0$ and $N = n$.

1.6. **Organization.** In Section 2, we prepare some notations and recall the basic facts for Riemannian manifolds with boundary. In Section 3, we show Laplacian comparison theorems for the distance function from

the boundary. In Section 4, we prove Theorem 1.1. In Section 5, we show several volume comparison theorems, and conclude Theorems 1.2 and 1.3. In Section 6, we prove Theorems 1.4 and 1.5, and study the variants of the splitting theorems. In Section 7, we prove Theorems 1.6 and 1.7, and study explicit lower bounds for $\mu_{f,1,p}$. In Section 8, we prove Theorem 1.8.

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2. Preliminaries

We refer to [5] for the basics of metric geometry, and to [39] for the basics of Riemannian manifolds with boundary.

2.1. **Metric spaces.** Let (X, d_X) be a metric space with metric d_X. For $r > 0$ and $A \subset X$, we denote by $U_r(A)$ the open r-neighborhood of A in X, and by $B_r(A)$ the closed one. For $A_1, A_2 \subset X$, we put $d_X(A_1, A_2) := \inf_{x_1 \in A_1, x_2 \in A_2} d_X(x_1, x_2)$.

For a metric space (X, d_X), the length metric $\bar{d}_X$ is defined as follows: For two points $x_1, x_2 \in X$, we put $\bar{d}_X(x_1, x_2)$ to the infimum of the length of curves connecting x_1 and x_2 with respect to d_X. A metric space (X, d_X) is said to be a *length space* if $d_X = \bar{d}_X$.

Let (X, d_X) be a metric space. For an interval I, we say that a curve $\gamma : I \to X$ is a *normal minimal geodesic* if for all $s, t \in I$ we have $d_X(\gamma(s), \gamma(t)) = |s - t|$, and γ is a *normal geodesic* if for each $t \in I$ there exists an interval $J \subset I$ with $t \in J$ such that $\gamma|_J$ is a normal minimal geodesic. A metric space (X, d_X) is said to be a *geodesic space* if for every pair of points in X, there exists a normal minimal geodesic connecting them. A metric space is *proper* if all closed bounded subsets of the space are compact. The Hopf-Rinow theorem for length spaces states that if a length space (X, d_X) is complete and locally compact, and if $d_X < \infty$, then (X, d_X) is a proper geodesic space (see e.g., Theorem 2.5.23 in [5]).

2.2. **Riemannian manifolds with boundary.** For $n \geq 2$, let M be an n-dimensional, connected Riemannian manifold with (smooth) boundary with Riemannian metric g. For $p \in \operatorname{Int} M$, let T_pM be the tangent space at p on M, and let U_pM be the unit tangent sphere

at p on M. We denote by $\|\cdot\|$ the standard norm induced from g. If $v_1, \dots, v_k \in T_pM$ are linearly independent, then we see $\|v_1 \wedge \cdots \wedge v_k\| = \sqrt{\det(g(v_i, v_j))}$. Let d_M be the length metric induced from g. If M is complete with respect to d_M, then the Hopf-Rinow theorem for length spaces tells us that the metric space (M, d_M) is a proper geodesic space.

For $i = 1, 2$, let M_i be connected Riemannian manifolds with boundary with Riemannian metric g_i. For each i, the boundary ∂M_i carries the induced Riemannian metric h_i. We say that a homeomorphism $\Phi : M_1 \to M_2$ is a *Riemannian isometry with boundary* from M_1 to M_2 if Φ satisfies the following conditions:

(1) $\Phi|_{\operatorname{Int} M_1} : \operatorname{Int} M_1 \to \operatorname{Int} M_2$ is smooth, and $(\Phi|_{\operatorname{Int} M_1})^*(g_2) = g_1$;
(2) $\Phi|_{\partial M_1} : \partial M_1 \to \partial M_2$ is smooth, and $(\Phi|_{\partial M_1})^*(h_2) = h_1$.

If $\Phi : M_1 \to M_2$ is a Riemannian isometry with boundary, then the inverse Φ^{-1} is also a Riemannian isometry with boundary. Notice that there exists a Riemannian isometry with boundary from M_1 to M_2 if and only if the metric space (M_1, d_{M_1}) is isometric to (M_2, d_{M_2}) (see e.g., Section 2 in [40]).

2.3. **Jacobi fields orthogonal to the boundary.** Let M be a connected Riemannian manifold with boundary with Riemannian metric g. For a point $x \in \partial M$, and for the tangent space $T_x\partial M$ at x on ∂M, let $T_x^\perp \partial M$ be the orthogonal complement of $T_x\partial M$ in the tangent space at x on M. Take $u \in T_x^\perp \partial M$. For the second fundamental form S of ∂M, let $A_u : T_x\partial M \to T_x\partial M$ be the *shape operator* for u defined as

$$g(A_u v, w) := g(S(v, w), u).$$

We denote by u_x the unit inner normal vector at x. The *mean curvature* H_x at x is defined as $H_x := \operatorname{trace} A_{u_x}$. We denote by $\gamma_x : [0, T) \to M$ the normal geodesic with initial conditions $\gamma_x(0) = x$ and $\gamma_x'(0) = u_x$. We say that a Jacobi field Y along γ_x is a *∂M-Jacobi field* if Y satisfies the following initial conditions:

$$Y(0) \in T_x\partial M, \quad Y'(0) + A_{u_x} Y(0) \in T_x^\perp \partial M.$$

We say that $\gamma_x(t_0)$ is a *conjugate point* of ∂M along γ_x if there exists a non-zero ∂M-Jacobi field Y along γ_x with $Y(t_0) = 0$. We denote by $\tau_1(x)$ the first conjugate value for ∂M along γ_x. It is well-known that for all $x \in \partial M$ and $t > \tau_1(x)$, we have $t > \rho_{\partial M}(\gamma_x(t))$.

For a point $x \in \partial M$, and for a piecewise smooth vector field X along γ_x with $X(0) \in T_x\partial M$, the *index form* of γ_x is defined as

$$I_{\partial M}(X, X) := \int_0^t g(X'(t), X'(t)) - g(R(X(t), \gamma_x'(t))\gamma_x'(t), X(t))\, dt \\ - g(A_{u_x} X(0), X(0)).$$

Lemma 2.1. *For $x \in \partial M$, we suppose that there exists no conjugate point of ∂M on $\gamma_x|_{[0,t_0]}$. Then for every piecewise smooth vector field X along γ_x with $X(0) \in T_x\partial M$, there exists a unique ∂M-Jacobi field Y along γ_x with $X(t_0) = Y(t_0)$ such that*

$$I_{\partial M}(Y, Y) \leq I_{\partial M}(X, X);$$

the equality holds if and only if $X = Y$ on $[0, t_0]$.

For the normal tangent bundle $T^\perp\partial M := \bigcup_{x\in\partial M} T_x^\perp\partial M$ of ∂M, let $0(T^\perp\partial M)$ be the zero-section $\bigcup_{x\in\partial M}\{\, 0_x \in T_x^\perp\partial M \,\}$ of $T^\perp\partial M$. On an open neighborhood of $0(T^\perp\partial M)$ in $T^\perp\partial M$, the normal exponential map $\exp^\perp$ of ∂M is defined as $\exp^\perp(x, u) := \gamma_x(\|u\|)$ for $x \in \partial M$ and $u \in T_x^\perp\partial M$.

For $x \in \partial M$ and $t \in [0, \tau_1(x))$, we denote by $\theta(t, x)$ the absolute value of the Jacobian of $\exp^\perp$ at $(x, tu_x) \in T^\perp\partial M$. For each $x \in \partial M$, we choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. For each i, let $Y_{x,i}$ be the ∂M-Jacobi field along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y_{x,i}'(0) = -A_{u_x} e_{x,i}$. Note that for all $x \in \partial M$ and $t \in [0, \tau_1(x))$, we have $\theta(t, x) = \|Y_{x,1}(t) \wedge \cdots \wedge Y_{x,n-1}(t)\|$. This does not depend on the choice of the orthonormal bases.

2.4. **Cut locus for the boundary.** We recall the basic properties of the cut locus for the boundary. The basic properties seem to be well-known. We refer to [40] for the proofs.

Let M be a connected complete Riemannian manifold with boundary with Riemannian metric g. For $p \in M$, we call $x \in \partial M$ a *foot point* on ∂M of p if $d_M(p, x) = \rho_{\partial M}(p)$. Since (M, d_M) is proper, every point in M has at least one foot point on ∂M. For $p \in \operatorname{Int} M$, let $x \in \partial M$ be a foot point on ∂M of p. Then there exists a unique normal minimal geodesic $\gamma : [0, l] \to M$ from x to p such that $\gamma = \gamma_x|_{[0,l]}$, where $l = \rho_{\partial M}(p)$. In particular, $\gamma'(0) = u_x$ and $\gamma|_{(0,l]}$ lies in $\operatorname{Int} M$.

Let $\tau : \partial M \to \mathbb{R} \cup \{\infty\}$ be the function defined as (1.5). By the property of τ_1, for all $x \in \partial M$ we have $0 < \tau(x) \leq \tau_1(x)$. The function τ is continuous on ∂M.

We have already known the following (see e.g., Section 3 in [40]):

Proposition 2.2. *For every $r \in (0,\infty)$ we have*

$$B_r(\partial M) = \exp^{\perp}\left(\bigcup_{x\in\partial M} \{tu_x \mid t \in [0, \min\{r, \tau(x)\}]\} \right).$$

For the inscribed radius $D(M, \partial M)$ of M, from the definition of τ, we deduce $D(M, \partial M) = \sup_{x\in\partial M} \tau(x)$.

The continuity of τ implies the following (see e.g., Section 3 in [40]):

Lemma 2.3. *Suppose that ∂M is compact. Then $D(M, \partial M)$ is finite if and only if M is compact.*

We put

$$TD_{\partial M} := \bigcup_{x\in\partial M} \{\, t\, u_x \in T^{\perp}_x \partial M \mid t \in [0, \tau(x))\,\},$$

$$T\mathrm{Cut}\,\partial M := \bigcup_{x\in\partial M} \{\, \tau(x)\, u_x \in T^{\perp}_x \partial M \mid \tau(x) < \infty\,\},$$

and define $D_{\partial M} := \exp^{\perp}(TD_{\partial M})$ and $\mathrm{Cut}\,\partial M := \exp^{\perp}(T\mathrm{Cut}\,\partial M)$. We call $\mathrm{Cut}\,\partial M$ the *cut locus for the boundary* ∂M. By the continuity of τ, the set $\mathrm{Cut}\,\partial M$ is a null set of M. Furthermore, we have

$$\mathrm{Int}\, M = (D_{\partial M} \setminus \partial M) \sqcup \mathrm{Cut}\,\partial M, \quad M = D_{\partial M} \sqcup \mathrm{Cut}\,\partial M.$$

This implies that if $\mathrm{Cut}\,\partial M = \emptyset$, then ∂M is connected. The set $TD_{\partial M} \setminus 0(T^{\perp}\partial M)$ is a maximal domain in $T^{\perp}\partial M$ on which $\exp^{\perp}$ is regular and injective.

The following has been shown in the proof of Theorem 1.3 in [40]:

Lemma 2.4. *If there exists a connected component ∂M_0 of ∂M such that for all $x \in \partial M_0$ we have $\tau(x) = \infty$, then ∂M is connected and* $\mathrm{Cut}\,\partial M = \emptyset$.

The function $\rho_{\partial M}$ is smooth on $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$. For each $p \in \mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$, the gradient vector $\nabla \rho_{\partial M}(p)$ of $\rho_{\partial M}$ at p is given by $\nabla \rho_{\partial M}(p) = \gamma'(l)$, where $\gamma : [0, l] \to M$ is the normal minimal geodesic from the foot point on ∂M of p to p.

For $\Omega \subset M$, we denote by $\bar{\Omega}$ the closure of Ω in M, and by $\partial\Omega$ the boundary of Ω in M. For a domain Ω in M such that $\partial\Omega$ is a smooth hypersurface in M, we denote by $\mathrm{vol}_{\partial\Omega}$ the canonical Riemannian volume measure on $\partial\Omega$.

We have the following fact to avoid the cut locus for the boundary:

Lemma 2.5. *Let Ω be a domain in M such that $\partial\Omega$ is a smooth hypersurface in M. Then there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of closed subsets of $\bar{\Omega}$ satisfying that for every $k \in \mathbb{N}$, the set $\partial\Omega_k$ is a smooth hypersurface*

in M except for a null set in $(\partial\Omega, \mathrm{vol}_{\partial\Omega})$, and satisfying the following properties:

(1) *for all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$;*
(2) $\bar{\Omega} \setminus \mathrm{Cut}\,\partial M = \bigcup_{k\in\mathbb{N}} \Omega_k$*;*
(3) *for every $k \in \mathbb{N}$, and for almost every point $p \in \partial\Omega_k \cap \partial\Omega$ in $(\partial\Omega, \mathrm{vol}_{\partial\Omega})$, there exists the unit outer normal vector for Ω_k at p that coincides with the unit outer normal vector on $\partial\Omega$ for Ω at p;*
(4) *for every $k \in \mathbb{N}$, on $\partial\Omega_k \setminus \partial\Omega$, there exists the unit outer normal vector field ν_k for Ω_k such that $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.*

Moreover, if $\bar{\Omega} = M$, then for every $k \in \mathbb{N}$, the set $\partial\Omega_k$ is a smooth hypersurface in M, and satisfies $\partial\Omega_k \cap \partial M = \partial M$.

For the cut locus for a single point, a similar result to Lemma 2.5 is well-known (see e.g., Theorem 4.1 in [9]). One can prove Lemma 2.5 by a similar method to that of the proof of the result for the cut locus for a single point. We omit the proof.

2.5. **Busemann functions and asymptotes.** Let M be a connected complete Riemannian manifold with boundary. A normal geodesic $\gamma : [0,\infty) \to M$ is said to be a *ray* if for all $s, t \in [0,\infty)$ it holds that $d_M(\gamma(s), \gamma(t)) = |s - t|$. For a ray $\gamma : [0,\infty) \to M$, the *Busemann function $b_\gamma : M \to \mathbb{R}$ of γ* is defined as

$$b_\gamma(p) := \lim_{t\to\infty} (t - d_M(p, \gamma(t))).$$

Take a ray $\gamma : [0,\infty) \to M$ and a point $p \in \mathrm{Int}\, M$, and choose a sequence $\{t_i\}$ with $t_i \to \infty$. For each i, we take a normal minimal geodesic $\gamma_i : [0, l_i] \to M$ from p to $\gamma(t_i)$. Since γ is a ray, it follows that $l_i \to \infty$. Take a sequence $\{T_j\}$ with $T_j \to \infty$. Using the fact that M is proper, we take a subsequence $\{\gamma_{1,i}\}$ of $\{\gamma_i\}$, and a normal minimal geodesic $\gamma_{p,1} : [0, T_1] \to M$ from p to $\gamma_{p,1}(T_1)$ such that $\gamma_{1,i}|_{[0,T_1]}$ uniformly converges to $\gamma_{p,1}$. In this manner, take a subsequence $\{\gamma_{2,i}\}$ of $\{\gamma_{1,i}\}$ and a normal minimal geodesic $\gamma_{p,2} : [0, T_2] \to M$ from p to $\gamma_{p,2}(T_2)$ such that $\gamma_{2,i}|_{[0,T_2]}$ uniformly converges to $\gamma_{p,2}$, where $\gamma_{p,2}|_{[0,T_1]} = \gamma_{p,1}$. By means of a diagonal argument, we obtain a subsequence $\{\gamma_k\}$ of $\{\gamma_i\}$ and a ray γ_p in M such that for every $t \in (0,\infty)$ we have $\gamma_k(t) \to \gamma_p(t)$ as $k \to \infty$. We call such a ray γ_p an *asymptote for γ from p*.

The following lemmas have been shown in [40].

Lemma 2.6. *Suppose that for some $x \in \partial M$ we have $\tau(x) = \infty$. Take $p \in \mathrm{Int}\, M$. If $b_{\gamma_x}(p) = \rho_{\partial M}(p)$, then $p \notin \mathrm{Cut}\,\partial M$. Moreover, for the unique foot point y on ∂M of p, we have $\tau(y) = \infty$.*

Lemma 2.7. *Suppose that for some $x \in \partial M$ we have $\tau(x) = \infty$. For $l \in (0, \infty)$, put $p := \gamma_x(l)$. Then there exists $\epsilon \in (0, \infty)$ such that for all $q \in B_\epsilon(p)$, all asymptotes for the ray γ_x from q lie in* $\operatorname{Int} M$.

2.6. **Weighted Laplacians.** Let M be a connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function. For a smooth function ϕ on M, the *weighted Laplacian $\Delta_f \phi$ for ϕ* is defined by

$$\Delta_f \phi := \Delta \phi + g(\nabla f, \nabla \phi),$$

where $\Delta \phi$ is the Laplacian for ϕ defined as the minus of the trace of its Hessian. Notice that Δ_f coincides with the $(f, 2)$-Laplacian $\Delta_{f,2}$.

For $x \in \partial M$ and $t \in [0, \tau_1(x))$, we put $\theta_f(t, x) := e^{-f(\gamma_x(t))}\,\theta(t, x)$. For all $x \in \partial M$ and $t \in (0, \tau(x))$, we see

$$\Delta_f\, \rho_{\partial M}(\gamma_x(t)) = -(\log \theta(t, x))' + f(\gamma_x(t))' = -\frac{\theta_f'(t, x)}{\theta_f(t, x)}. \tag{2.1}$$

For $\kappa \in \mathbb{R}$, let $s_\kappa(t)$ be a unique solution of the so-called Jacobi-equation $\phi''(t) + \kappa\phi(t) = 0$ with initial conditions $\phi(0) = 0$ and $\phi'(0) = 1$. We put $c_\kappa(t) := s_\kappa'(t)$.

For $p \in M$, let $\rho_p : M \to \mathbb{R}$ denote the distance function from p defined as $\rho_p(q) := d_M(p, q)$.

Qian [38] has proved a Laplacian comparison inequality for the distance function from a single point (see equation 7 in [38]). In our setting, the comparison inequality holds in the following form:

Lemma 2.8 ([38])**.** *For $N \in [n, \infty)$, we suppose $\operatorname{Ric}^N_{f,M} \geq (N-1)\kappa$. Take $p \in \operatorname{Int} M$. Assume that there exists $q \in \operatorname{Int} M \setminus \{p\}$ such that a normal minimal geodesic in M from p to q lies in $\operatorname{Int} M$, and ρ_p is smooth at q. Then*

$$\Delta_f\, \rho_p(q) \geq -(N-1)\frac{c_\kappa(\rho_p(q))}{s_\kappa(\rho_p(q))}. \tag{2.2}$$

Remark 2.1. In Lemma 2.8, we choose a normal minimal geodesic $\gamma : [0, l] \to M$ from p to q that lies in $\operatorname{Int} M$, and an orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM with $e_n = \gamma'(0)$. Let $\{Y_i\}_{i=1}^{n-1}$ be the Jacobi fields along γ with initial conditions $Y_i(0) = 0$ and $Y_i'(0) = e_i$. If the equality in (2.2) holds, then for all i we see $Y_i = s_\kappa\, E_i$ on $[0, l]$, where $\{E_i\}_{i=1}^{n-1}$ are the parallel vector fields along γ with initial condition $E_i(0) = e_i$.

Remark 2.2. Kasue and Kumura [23] have been proved Lemma 2.8 in the case where N is an integer, and $\kappa \leq 0$.

Let $\phi : M \to \mathbb{R}$ be a continuous function, and let U be a domain contained in $\operatorname{Int} M$. For $p \in U$, and for a function ψ defined on an open neighborhood of p, we say that ψ is a *support function of ϕ at p* if we have $\psi(p) = \phi(p)$ and $\psi \leq \phi$. We say that ϕ is *f-subharmonic on U* if for every $p \in U$, and for every $\epsilon \in (0, \infty)$, there exists a smooth, support function $\psi_{p,\epsilon}$ of ϕ at p such that $\Delta_f \, \psi_{p,\epsilon}(p) \leq \epsilon$.

We recall the following maximal principle of Calabi type (see e.g., [6], and Lemma 2.4 in [12]).

Lemma 2.9. *If an f-subharmonic function on a domain U contained in $\operatorname{Int} M$ takes the maximal value at a point in U, then it must be constant on U.*

Fang, Li and Zhang [12] have proved a subharmonicity of Busemann functions on manifolds without boundary (see Lemma 2.1 in [12]). In our setting, the subharmonicity holds in the following form:

Lemma 2.10 ([12])**.** *Assume $\sup f(M) < \infty$. Suppose $\operatorname{Ric}^{\infty}_{f,M} \geq 0$. Let $\gamma : [0, \infty) \to M$ be a ray that lies in $\operatorname{Int} M$, and let U be a domain contained in $\operatorname{Int} M$ such that for each $p \in U$, there exists an asymptote for γ from p that lies in $\operatorname{Int} M$. Then b_γ is f-subharmonic on U.*

3. Laplacian comparisons

In this section, let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function.

3.1. Basic comparisons.

We prove the following basic lemma:

Lemma 3.1. *Take $x \in \partial M$. For $N \in [n, \infty)$, suppose that for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$ we have $\operatorname{Ric}^N_f(\gamma'_x(t)) \geq (N-1)\kappa$, and suppose $H_{f,x} \geq (N-1)\lambda$. Then for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$ we have*

$$\frac{\theta'_f(t,x)}{\theta_f(t,x)} \leq (N-1) \frac{s'_{\kappa,\lambda}(t)}{s_{\kappa,\lambda}(t)}, \tag{3.1}$$

and for all $s, t \in [0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$ with $s \leq t$ we have

$$\frac{\theta_f(t,x)}{\theta_f(s,x)} \leq \frac{s^{N-1}_{\kappa,\lambda}(t)}{s^{N-1}_{\kappa,\lambda}(s)}; \tag{3.2}$$

in particular, $\theta_f(t,x) \leq e^{-f(x)} \, s^{N-1}_{\kappa,\lambda}(t)$.

Proof. Put $F := f \circ \gamma_x$. From direct computations, it follows that

$$\frac{\theta'_f(t,x)}{\theta_f(t,x)} = \frac{\theta'(t,x)}{\theta(t,x)} - F'(t) \tag{3.3}$$

for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$. Choose an orthonormal basis $\{e_i\}_{i=1}^{n-1}$ of $T_x\partial M$. For each i, we denote by E_i the parallel vector field along γ_x with initial condition $E_i(0) = e_i$. We fix $t_0 \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$, and put $W_i(t) := (s_{\kappa,\lambda}(t)/s_{\kappa,\lambda}(t_0))E_i(t)$ for $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$. For a unique ∂M-Jacobi field $Y_{t_0,i}$ along $\gamma_x|_{[0,t_0]}$ with initial conditions $Y_{t_0,i}(t_0) = W_i(t_0)\,(= E_i(t_0))$ and $Y'_{t_0,i}(0) = -A_{u_x} Y_{t_0,i}(0)$, let $\theta_{t_0}(t) := \|Y_{t_0,1}(t) \wedge \cdots \wedge Y_{t_0,n-1}(t)\|$ for $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$. The linearity of the Jacobi equations implies that for the ∂M-Jacobi field Y_i along γ_x with initial conditions $Y_i(0) = e_i$ and $Y_i'(0) = -A_{u_x} Y_i(0)$, there exist some constants $\{a_{ij}\}_{j=1}^{n-1}$ satisfying $Y_i = \sum_{j=1}^{n-1} a_{ij}\, Y_{t_0,j}$. Since $\theta_{t_0}(t_0) = 1$, we have $\theta'(t_0,x)/\theta(t_0,x) = \theta'_{t_0}(t_0)$. Furthermore,

$$\theta'_{t_0}(t_0) = \sum_{i=1}^{n-1} g(Y_{t_0,i}(t_0), Y'_{t_0,i}(t_0)) = \sum_{i=1}^{n-1} I_{\partial M}(Y_{t_0,i}, Y_{t_0,i}). \tag{3.4}$$

We have $Y_{t_0,i}(t_0) = W_i(t_0)$. Therefore, Lemma 2.1 implies

$$\sum_{i=1}^{n-1} I_{\partial M}(Y_{t_0,i}, Y_{t_0,i}) \le \sum_{i=1}^{n-1} I_{\partial M}(W_i, W_i). \tag{3.5}$$

We assume $N > n$. Put $\phi(t) := \|W_i(t)\| (= s_{\kappa,\lambda}(t)/s_{\kappa,\lambda}(t_0))$ for $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$. Note that we have $\phi'(t) = \|W_i'(t)\|$ for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$. From (3.3), (3.4) and (3.5), it follows that

$$\begin{aligned}
\frac{\theta'_f(t_0,x)}{\theta_f(t_0,x)} &\le (n-1)\int_0^{t_0} \phi'(t)^2\,dt - \int_0^{t_0} \mathrm{Ric}_g(\gamma_x'(t))\,\phi(t)^2\,dt \\
&\qquad\qquad - H_x\,\phi(0)^2 - F'(t_0) \\
&= (N-1)\int_0^{t_0} \phi'(t)^2\,dt - \int_0^{t_0} \mathrm{Ric}_f^N(\gamma_x'(t))\,\phi(t)^2\,dt - H_{f,x}\phi(0)^2 \\
&\quad - (N-n)\int_0^{t_0} \phi'(t)^2\,dt + \int_0^{t_0}\left(F''(t) - \frac{1}{N-n}F'(t)^2\right)\phi(t)^2\,dt \\
&\qquad\qquad + F'(0)\,\phi(0)^2 - F'(t_0).
\end{aligned}$$

From the curvature assumptions, we derive

$$\begin{aligned}
\frac{\theta'_f(t_0,x)}{\theta_f(t_0,x)} &\le (N-1)\frac{s'_{\kappa,\lambda}(t_0)}{s_{\kappa,\lambda}(t_0)} - (N-n)\int_0^{t_0} \phi'(t)^2\,dt \\
&\quad + \int_0^{t_0}\left(F''(t) - \frac{1}{N-n}F'(t)^2\right)\phi(t)^2\,dt + F'(0)\,\phi(0)^2 - F'(t_0).
\end{aligned} \tag{3.6}$$

By integration by parts, we have

(3.7)
$$\int_0^{t_0} F''(t)\,\phi(t)^2\,dt = F'(t_0) - F'(0)\,\phi(0)^2 - 2\int_0^{t_0} F'(t)\phi'(t)\phi(t)\,dt.$$

Furthermore, for all $t \in (0, t_0)$, we have

$$\begin{aligned}(N-n)\phi'(t)^2 + 2F'(t)\phi'(t)\phi(t) + \frac{F'(t)^2\phi(t)^2}{N-n} \\ = \frac{\phi(t)^2}{N-n}\left((N-n)\frac{s'_{\kappa,\lambda}(t)}{s_{\kappa,\lambda}(t)} + F'(t)\right)^2 \geq 0.\end{aligned} \tag{3.8}$$

By using (3.6), (3.7) and (3.8), we obtain (3.1).

We assume $N = n$. In this case, f is a constant function; in particular, $H_{f,x} = H_x$ and $F'(t_0) = 0$. By Lemma 2.1, we see

$$\frac{\theta'_f(t_0,x)}{\theta_f(t_0,x)} \leq (n-1)\int_0^{t_0} \phi'(t)^2\,dt - \int_0^{t_0} \mathrm{Ric}^n_f(\gamma'_x(t))\,\phi(t)^2\,dt - H_{f,x}\,\phi(0)^2.$$

The curvature assumptions imply (3.1).

By (3.1), for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$, we have

$$\frac{d}{dt}\log\frac{s^{N-1}_{\kappa,\lambda}(t)}{\theta_f(t,x)} = (N-1)\frac{s'_{\kappa,\lambda}(t)}{s_{\kappa,\lambda}(t)} - \frac{\theta'_f(t,x)}{\theta_f(t,x)} \geq 0.$$

This implies the inequality (3.2). □

In [17], Lemma 3.1 has been proved when $f = 0$ and $N = n$.

Remark 3.1. In Lemma 3.1, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$, and let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x}e_{x,i}$. Suppose that for some $t_0 \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$ the equality in (3.1) holds. Then the equality in (3.5) also holds. By Lemma 2.1, for all i we have $Y_{x,i} = s_{\kappa,\lambda}\,E_{x,i}$ on $[0, t_0]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, if $N > n$, then the equality in (3.8) holds on $[0, t_0]$. This implies $f \circ \gamma_x = f(x) - (N-n)\log s_{\kappa,\lambda}$ on $[0, t_0]$.

In the case of $N = \infty$, we have the following:

Lemma 3.2. *Take $x \in \partial M$. Suppose that for all $t \in (0, \tau_1(x))$ we have $\mathrm{Ric}^\infty_f(\gamma'_x(t)) \geq 0$, and suppose $H_{f,x} \geq 0$. Then for all $t \in (0, \tau_1(x))$, we have $\theta'_f(t,x) \leq 0$. In particular, for all $s, t \in [0, \tau_1(x))$ with $s \leq t$, we have $\theta_f(t,x) \leq \theta_f(s,x)$.*

Proof. Let $F := f \circ \gamma_x$. Choose an orthonormal basis $\{e_i\}_{i=1}^{n-1}$ of $T_x\partial M$. For each i, let E_i denote the parallel vector field along γ_x with initial

condition $E_i(0) = e_i$. Put $\phi(t) := \|E_i(t)\|(= 1)$ for $t \in (0, \tau_1(x))$. Fix $t_0 \in (0, \tau_1(x))$. By Lemma 2.1, we see

$$\frac{\theta'_f(t_0,x)}{\theta_f(t_0,x)} \leq -\int_0^{t_0} \left(\mathrm{Ric}_f^\infty(\gamma'_x(t)) - F''(t)\right) \phi(t)^2 \, dt \\ - (H_{f,x} - F'(0))\, \phi(0)^2 - F'(t_0).$$

By the curvature assumptions, and by integration by parts, we have

$$\theta'_f(t_0,x) \leq \theta_f(t_0,x) \left(\int_0^{t_0} F''(t)\, \phi(t)^2 \, dt + F'(0)\, \phi(0)^2 - F'(t_0) \right) = 0.$$

This proves the lemma. □

Remark 3.2. In Lemma 3.2, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$, and let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. Suppose that for some $t_0 \in (0, \tau_1(x))$ we have $\theta'_f(t_0, x) = 0$. By Lemma 2.1, for all i we have $Y_{x,i} = E_{x,i}$ on $[0, t_0]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$.

3.2. **Laplacian comparisons.** Combining Lemma 3.1 and (2.1), we have the following Laplacian comparison result:

Lemma 3.3. *Take $x \in \partial M$. For $N \in [n, \infty)$, we suppose that for all $t \in (0, \tau(x))$ we have $\mathrm{Ric}_f^N(\gamma'_x(t)) \geq (N-1)\kappa$, and suppose $H_{f,x} \geq (N-1)\lambda$. Then for all $t \in (0, \tau(x))$ we have*

$$\Delta_f \, \rho_{\partial M}(\gamma_x(t)) \geq -(N-1) \frac{s'_{\kappa,\lambda}(t)}{s_{\kappa,\lambda}(t)}.$$

In [19], Lemma 3.3 has been proved when $f = 0$ and $N = n$.
In the case of $N = \infty$, by using Lemma 3.2 and (2.1), we have:

Lemma 3.4. *Take $x \in \partial M$. Suppose that for all $t \in (0, \tau(x))$ we have $\mathrm{Ric}_f^\infty(\gamma'_x(t)) \geq 0$, and suppose $H_{f,x} \geq 0$. Then for all $t \in (0, \tau(x))$ we have $\Delta_f \rho_{\partial M}(\gamma_x(t)) \geq 0$.*

Remark 3.3. The equality case in Lemma 3.3 (resp. 3.4) results into that in Lemma 3.1 (resp. 3.2) (see Remarks 3.1 and 3.2).

3.3. **Distributions.** From Lemma 3.3, we derive the following:

Lemma 3.5. *Take $x \in \partial M$. Let $p \in (1, \infty)$. For $N \in [n, \infty)$, we suppose that for all $t \in (0, \tau(x))$ we have $\mathrm{Ric}_f^N(\gamma'_x(t)) \geq (N-1)\kappa$, and suppose $H_{f,x} \geq (N-1)\lambda$. Let $\phi : [0, \infty) \to \mathbb{R}$ be a monotone increasing*

smooth function. Then for all $t \in (0, \tau(x))$ *we have*

$$\Delta_{f,p}\,(\phi \circ \rho_{\partial M})(\gamma_x(t)) \geq -\left((\phi')^{p-1}\right)'(t) - (N-1)\frac{s'_{\kappa,\lambda}(t)}{s_{\kappa,\lambda}(t)}\,\phi'(t)^{p-1}. \tag{3.9}$$

Proof. By straightforward computations, for all $t \in (0, \tau(x))$

$$\Delta_{f,p}\,(\phi \circ \rho_{\partial M})(\gamma_x(t)) = -\left((\phi')^{p-1}\right)'(t) + \Delta_{f,2}\,\rho_{\partial M}(\gamma_x(t))\,\phi'(t)^{p-1}.$$

This together with Lemma 3.3, we obtain (3.9). □

In the case of $N = \infty$, we have:

Lemma 3.6. *Take* $x \in \partial M$. *Let* $p \in (1, \infty)$. *Suppose that for all* $t \in (0, \tau(x))$ *we have* $\mathrm{Ric}^{\infty}_f(\gamma'_x(t)) \geq 0$, *and suppose* $H_{f,x} \geq 0$. *Let* $\phi : [0, \infty) \to \mathbb{R}$ *be a monotone increasing smooth function. Then for all* $t \in (0, \tau(x))$

$$\Delta_{f,p}(\phi \circ \rho_{\partial M})(\gamma_x(t)) \geq -\left((\phi')^{p-1}\right)'(t). \tag{3.10}$$

Proof. For all $t \in (0, \tau(x))$, we have

$$\Delta_{f,p}\,(\phi \circ \rho_{\partial M})(\gamma_x(t)) = -\left((\phi')^{p-1}\right)'(t) + \Delta_{f,2}\,\rho_{\partial M}(\gamma_x(t))\,\phi'(t)^{p-1}.$$

Lemma 3.4 implies (3.10). □

Remark 3.4. The equality case in Lemma 3.5 (resp. 3.6) results into that in Lemma 3.3 (resp. 3.4) (see Remarks 3.1, 3.2 and 3.3).

By Lemma 3.5, we have the following:

Proposition 3.7. *Let* $p \in (1, \infty)$. *For* $N \in [n, \infty)$, *we suppose* $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ *and* $H_{f,\partial M} \geq (N-1)\lambda$. *For a monotone increasing smooth function* $\phi : [0, \infty) \to \mathbb{R}$, *we put* $\Phi := \phi \circ \rho_{\partial M}$. *Then we have*

$$\Delta_{f,p}\,\Phi \geq \left(-\left((\phi')^{p-1}\right)' - (N-1)\frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}\,(\phi')^{p-1}\right) \circ \rho_{\partial M}$$

in a distribution sense on M. *More precisely, for every non-negative smooth function* $\psi : M \to \mathbb{R}$ *whose support is compact and contained in* $\mathrm{Int}\, M$, *we have*

$$\begin{aligned} &\int_M \|\nabla \Phi\|^{p-2} g\,(\nabla \psi, \nabla \Phi)\; d\,m_f \\ &\geq \int_M \psi\,\left(\left(-\left((\phi')^{p-1}\right)' - (N-1)\frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}\,(\phi')^{p-1}\right) \circ \rho_{\partial M}\right) d\,m_f. \end{aligned} \tag{3.11}$$

Proof. By Lemma 2.5, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of closed subsets of M satisfying that for every k, the set $\partial\Omega_k$ is a smooth hypersurface in M, and satisfying the following: (1) for all k_1, k_2 with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $M \setminus \operatorname{Cut}\partial M = \bigcup_k \Omega_k$; (3) $\partial\Omega_k \cap \partial M = \partial M$ for all k; (4) for each k, on $\partial\Omega_k \setminus \partial M$, there exists the unit outer normal vector field ν_k for Ω_k with $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.

For the canonical Riemannian volume measure vol_k on $\partial\Omega_k \setminus \partial M$, put $m_{f,k} := e^{-f|_{\partial\Omega_k\setminus\partial M}} \mathrm{vol}_k$. Let $\psi : M \to \mathbb{R}$ be a non-negative smooth function whose support is compact and contained in $\operatorname{Int} M$. By the Green formula, and by $\partial\Omega_k \cap \partial M = \partial M$, we have

$$\begin{aligned}
&\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right)\, d\,m_f \\
&= \int_{\Omega_k} \left(-\psi\, g\left(\nabla\left(\|\nabla\Phi\|^{p-2}\right), \nabla\Phi\right) + \|\nabla\Phi\|^{p-2}\,\psi\,\Delta_{f,2}\Phi\right)\, d\,m_f \\
&\qquad\qquad + \int_{\partial\Omega_k\setminus\partial M} \|\nabla\Phi\|^{p-2}\,\psi\, g\left(\nu_k, \nabla\Phi\right)\, d\,m_{f,k} \\
&= \int_{\Omega_k} \psi\,\Delta_{f,p}\Phi\, d\,m_f + \int_{\partial\Omega_k\setminus\partial M} \|\nabla\Phi\|^{p-2}\,\psi\, g\left(\nu_k, \nabla\Phi\right)\, d\,m_{f,k}.
\end{aligned}$$

Lemma 3.5 and $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$ imply

$$\begin{aligned}
&\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right)\, d\,m_f \\
&\quad \geq \int_{\Omega_k} \psi\, \left(\left(-\left((\phi')^{p-1}\right)' - (N-1)\frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}\,(\phi')^{p-1}\right)\circ\rho_{\partial M}\right)\, d\,m_f.
\end{aligned}$$

Letting $k \to \infty$, we obtain the desired inequality. $\square$

In the case of $N = \infty$, we have:

Proposition 3.8. *Let $p \in (1,\infty)$. Suppose $\operatorname{Ric}^{\infty}_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. For a monotone increasing smooth function $\phi : [0,\infty) \to \mathbb{R}$, put $\Phi := \phi \circ \rho_{\partial M}$. Then we have*

$$\Delta_{f,p}\,\Phi \geq -\left((\phi')^{p-1}\right)' \circ \rho_{\partial M}$$

in a distribution sense on M. More precisely, for every non-negative smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in $\operatorname{Int} M$, we have

$$\int_M \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right)\, d\,m_f \geq \int_M \psi\,\left(-\left((\phi')^{p-1}\right)' \circ \rho_{\partial M}\right)\, d\,m_f. \tag{3.12}$$

Proof. Lemma 2.5 implies that there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of closed subsets of M satisfying that for every k, the set $\partial\Omega_k$ is a smooth hypersurface in M, and satisfying the following: (1) for all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $M \setminus \mathrm{Cut}\,\partial M = \bigcup_k \Omega_k$; (3) $\partial\Omega_k \cap \partial M = \partial M$ for all k; (4) for each k, on $\partial\Omega_k \setminus \partial M$, there exists the unit outer normal vector field ν_k for Ω_k with $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.

For the canonical Riemannian volume measure vol_k on $\partial\Omega_k \setminus \partial M$, put $m_{f,k} := e^{-f|_{\partial\Omega_k\setminus\partial M}}\,\mathrm{vol}_k$. Let $\psi : M \to \mathbb{R}$ be a non-negative smooth function whose support is compact and contained in $\mathrm{Int}\,M$. By the Green formula, and by $\partial\Omega_k \cap \partial M = \partial M$, we see

$$\begin{aligned}\int_{\Omega_k} &\|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right)\, d\,m_f \\ &= \int_{\Omega_k} \psi\, \Delta_{f,p}\Phi\, d\,m_f + \int_{\partial\Omega_k\setminus\partial M} \|\nabla\Phi\|^{p-2}\,\psi\, g\left(\nu_k, \nabla\Phi\right)\, d\,m_{f,k}.\end{aligned}$$

By Lemma 3.5 and $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$,

$$\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right)\, d\,m_f \geq \int_{\Omega_k} \psi\, \left(-\left(\left(\phi'\right)^{p-1}\right)' \circ \rho_{\partial M}\right)\, d\,m_f.$$

By letting $k \to \infty$, we complete the proof. □

Remark 3.5. In Proposition 3.7 (resp. 3.8), assume that the equality in (3.11) (resp. (3.12)) holds. In this case, for a fixed $x \in \partial M$ we see that for every $t \in (0, \tau(x))$ the equality in (3.9) (resp. (3.10)) also holds. The equality case in Proposition 3.7 (resp. 3.8) results into that in Lemma 3.5 (resp. 3.6) (see Remark 3.4).

Remark 3.6. Perales [37] has proved a Laplacian comparison inequality for the distance function from the boundary in a barrier sense for manifolds with boundary of non-negative Ricci curvature. We can prove that the Laplacian comparison inequalities for $\rho_{\partial M}$ in Lemmas 3.3 and 3.4 globally hold on M in a barrier sense.

4. Inscribed radius rigidity

Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function.

4.1. **Inscribed radius comparison.** From Lemma 3.1, we derive the following comparison result for the inscribed radius.

Lemma 4.1. *Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then $D(M, \partial M) \leq C_{\kappa,\lambda}$.*

Proof. Take $x \in \partial M$. We suppose $C_{\kappa,\lambda} < \tau_1(x)$. By Lemma 3.1, for all $t \in [0, C_{\kappa,\lambda})$ we have $\theta_f(t,x) \leq e^{-f(x)} s_{\kappa,\lambda}^{N-1}(t)$. Letting $t \to C_{\kappa,\lambda}$, we have $\theta(C_{\kappa,\lambda}, x) = 0$; in particular, $\gamma_x(C_{\kappa,\lambda})$ is a conjugate point of ∂M along γ_x. This is a contradiction. Hence, we have $\tau_1(x) \leq C_{\kappa,\lambda}$. The relationship between τ and τ_1 implies $\tau(x) \leq C_{\kappa,\lambda}$. Since $D(M, \partial M)$ is equal to $\sup_{x \in \partial M} \tau(x)$, we have $D(M, \partial M) \leq C_{\kappa,\lambda}$. $\square$

In [20], Lemma 4.1 has been proved when $f = 0$ and $N = n$.

4.2. **Inscribed radius rigidity.** Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. By Lemma 4.1, we have $D(M, \partial M) \leq C_{\kappa,\lambda}$.

Take $p_0 \in M$ satisfying $\rho_{\partial M}(p_0) = C_{\kappa,\lambda}$. We put

$$\Omega := \{p \in \mathrm{Int}\, M \setminus \{p_0\} \mid \rho_{\partial M}(p) + \rho_{p_0}(p) = C_{\kappa,\lambda}\}.$$

Take a foot point x_{p_0} on ∂M of p_0, and the normal minimal geodesic $\gamma_0 : [0, C_{\kappa,\lambda}] \to M$ from x_{p_0} to p_0. Then for all $t \in (0, C_{\kappa,\lambda})$, we have $\gamma_0(t) \in \Omega$. Therefore, Ω is a non-empty closed subset of $\mathrm{Int}\, M \setminus \{p_0\}$.

We prove that Ω is an open subset of $\mathrm{Int}\, M \setminus \{p_0\}$. Fix $p \in \Omega$, and take a foot point x_p on ∂M of p. Note that x_p is also a foot point on ∂M of p_0. We take the normal minimal geodesic $\gamma : [0, C_{\kappa,\lambda}] \to M$ from x_p to p_0. Then $\gamma|_{(0, C_{\kappa,\lambda})}$ passes through p. There exists an open neighborhood U of p such that ρ_{p_0} and $\rho_{\partial M}$ are smooth on U, and for every $q \in U$ there exists a unique normal minimal geodesic in M from p_0 to q that lies in $\mathrm{Int}\, M$. By Lemmas 2.8 and 3.3, for all $q \in U$

$$\begin{aligned} \frac{\Delta_f(\rho_{\partial M} + \rho_{p_0})(q)}{N-1} &\geq -\left(\frac{\lambda c_\kappa(\rho_{\partial M}(q)) - \kappa s_\kappa(\rho_{\partial M}(q))}{c_\kappa(\rho_{\partial M}(q)) + \lambda s_\kappa(\rho_{\partial M}(q))} + \frac{c_\kappa(\rho_{p_0}(q))}{s_\kappa(\rho_{p_0}(q))} \right) \\ &= -\frac{s_{\kappa,\lambda}(\rho_{\partial M}(q) + \rho_{p_0}(q))}{s_{\kappa,\lambda}(\rho_{\partial M}(q)) s_\kappa(\rho_{p_0}(q))} \geq 0. \end{aligned}$$

Lemma 2.9 implies $U \subset \Omega$. We prove the openness of Ω.

Since $\mathrm{Int}\, M \setminus \{p_0\}$ is connected, we have $\Omega = \mathrm{Int}\, M \setminus \{p_0\}$, and hence $\rho_{\partial M} + \rho_{p_0} = C_{\kappa,\lambda}$ on M. This implies $M = B_{C_{\kappa,\lambda}}(p_0)$ and $\partial M = \partial B_{C_{\kappa,\lambda}}(p_0)$. Furthermore, we see that the cut locus for p_0 is empty, and the equality in (2.2) holds on $\mathrm{Int}\, M \setminus \{p_0\}$. For each $u \in U_{p_0} M$, choose an orthonormal basis $\{e_{u,i}\}_{i=1}^n$ of $T_{p_0} M$ with $e_n = u$. Let $\{Y_{u,i}\}_{i=1}^{n-1}$ be the Jacobi fields along γ_u with initial conditions $Y_{u,i}(0) = 0$ and $Y'_{u,i}(0) = e_{u,i}$, where $\gamma_u : [0, C_{\kappa,\lambda}] \to M$ is the normal geodesic with $\gamma_u(0) = p_0$ and $\gamma'_u(0) = u$. Then for all i we have $Y_{u,i} = s_\kappa\, E_{u,i}$ on $[0, C_{\kappa,\lambda}]$, where $\{E_{u,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_u with initial condition $E_{u,i}(0) = e_{u,i}$ (see Remark 2.1). Let $\tilde{p}_0$ denote the

center point of $B^n_{\kappa,\lambda}$. Choose a linear isometry $I : T_{p_0}M \to T_{\tilde{p}_0}B^n_{\kappa,\lambda}$. Define a map $\Phi : M \to B^n_{\kappa,\lambda}$ by $\Phi(p) := \exp_{\tilde{p}_0} \circ I \circ \exp^{-1}_{p_0}(p)$, where $\exp_{p_0}$ and $\exp_{\tilde{p}_0}$ are the exponential maps at p_0 and at $\tilde{p}_0$, respectively. For every $p \in \operatorname{Int} M$ the differential map $D(\Phi|_{\operatorname{Int} M})_p$ of $\Phi|_{\operatorname{Int} M}$ at p sends an orthonormal basis of T_pM to that of $T_{\Phi(p)}B^n_{\kappa,\lambda}$, and for every $x \in \partial M$ the map $D(\Phi|_{\partial M})_x$ sends an orthonormal basis of $T_x\partial M$ to that of $T_{\Phi(x)}\partial B^n_{\kappa,\lambda}$. Hence, Φ is a Riemannian isometry with boundary from M to $B^n_{\kappa,\lambda}$, and (M, d_M) is isometric to $(B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})$.

Now, the equality in Lemma 3.3 holds on $\operatorname{Int} M \setminus \{p_0\}$. For each $x \in \partial M$ we see $f \circ \gamma_x = f(x) - (N - n) \log s_{\kappa,\lambda}$ on $[0, C_{\kappa,\lambda}]$ (see Remark 3.3). If we suppose $N > n$, then $f(\gamma_x(t))$ tends to infinity as $t \to C_{\kappa,\lambda}$. This is a contradiction since $f(\gamma_x(C_{\kappa,\lambda})) = f(p_0)$. Hence, we obtain $N = n$. We complete the proof of Theorem 1.1. □

5. Volume comparisons

Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function.

5.1. **Absolute volume comparisons.** Let $\bar{\theta}_f : [0,\infty) \times \partial M \to \mathbb{R}$ be a function defined by

$$\bar{\theta}_f(t,x) := \begin{cases} \theta_f(t,x) & \text{if } t < \tau(x), \\ 0 & \text{if } t \geq \tau(x). \end{cases}$$

By the coarea formula (see e.g., Theorem 3.2.3 in [13]), we show:

Lemma 5.1. *Suppose that ∂M is compact. Then for all $r \in (0,\infty)$*

$$m_f(B_r(\partial M)) = \int_{\partial M} \int_0^r \bar{\theta}_f(t,x)\, dt\, d\operatorname{vol}_h, \tag{5.1}$$

where h is the induced Riemannian metric on ∂M.

Proof. Since ∂M is compact, $B_r(\partial M)$ is also compact; in particular, $m_f(B_r(\partial M)) < \infty$. From Proposition 2.2, we derive

$$B_r(\partial M) = \exp^{\perp}\left(\bigcup_{x \in \partial M} \{ t u_x \mid t \in [0, \min\{r, \tau(x)\}] \} \right).$$

The map $\exp^{\perp}$ is diffeomorphic on $\bigcup_{x\in\partial M}\{tu_x \mid t \in (0, \min\{r,\tau(x)\})\}$. Furthermore, the cut locus $\operatorname{Cut}\partial M$ for the boundary is a null set of M. Hence, the coarea formula and the Fubini theorem imply (5.1). □

Bayle [3] has stated the following absolute volume comparison inequality of Heintze-Karcher type without proof (see Theorem E.2.2 in [3], and also [35]).

Lemma 5.2 ([3]). *Suppose that ∂M is compact. For $N \in [n,\infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then for all $r \in (0,\infty)$*

$$m_f(B_r(\partial M)) \leq s_{N,\kappa,\lambda}(r)\, m_{f,\partial M}(\partial M); \tag{5.2}$$

in particular, we have (1.3).

Proof. Fix $r \in (0,\infty)$. By Lemma 3.1, for all $x \in \partial M$ and $t \in (0,r)$, we have $\bar{\theta}_f(t,x) \leq \bar{s}^{N-1}_{\kappa,\lambda}(t)\,\bar{\theta}_f(0,x)$. Integrate the both sides of the inequality over $(0,r)$ with respect to t, and then do that over ∂M with respect to x. By Lemma 5.1, we have (5.2). $\square$

Lemma 5.2 has been proved in [17] when $f=0$ and $N=n$.

In the case of $N=\infty$, Morgan [36] has shown the following volume comparison inequality (see Theorem 2 in [36], and also [35]).

Lemma 5.3 ([36]). *Suppose that ∂M is compact. Suppose $\mathrm{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. Then for all $r \in (0,\infty)$*

$$m_f(B_r(\partial M)) \leq r\, m_{f,\partial M}(\partial M); \tag{5.3}$$

in particular, we have (1.4).

Proof. Fix $r \in (0,\infty)$. By Lemma 3.2, for all $x \in \partial M$ and $t \in (0,r)$, we have $\bar{\theta}_f(t,x) \leq \bar{\theta}_f(0,x)$. Integrate the both sides of the inequality over $(0,r)$ with respect to t, and then do that over ∂M with respect to x. Lemma 5.1 implies the lemma. $\square$

Remark 5.1. In Lemma 5.2 (resp. 5.3), assume that for some $r>0$ the equality in (5.2) (resp. (5.3)) holds. For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0)=e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x}e_{x,i}$. Then for all i we see $Y_{x,i} = s_{\kappa,\lambda}\, E_{x,i}$ (resp. $Y_{x,i} = E_{x,i}$) on $[0,\min\{r,\bar{C}_{\kappa,\lambda}\}]$ (resp. $[0,r]$), where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0)=e_{x,i}$. Moreover, $f\circ\gamma_x = f(x) - (N-n)\log s_{\kappa,\lambda}$ on $[0,\min\{r,\bar{C}_{\kappa,\lambda}\}]$ (cf. Remarks 3.1 and 3.2).

5.2. **Relative volume comparison.** We have the following relative volume comparison theorem of Bishop-Gromov type:

Theorem 5.4. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. For $N \in [n,\infty)$, we suppose*

$\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ *and* $H_{f,\partial M} \geq (N-1)\lambda$. *Then for all* $r, R \in (0,\infty)$ *with* $r \leq R$, *we have*

$$\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} \leq \frac{s_{N,\kappa,\lambda}(R)}{s_{N,\kappa,\lambda}(r)}. \tag{5.4}$$

Proof. Lemma 3.1 implies that for all $s, t \in [0,\infty)$ with $s \leq t$,

$$\bar{\theta}_f(t,x)\ \bar{s}^{N-1}_{\kappa,\lambda}(s) \leq \bar{\theta}_f(s,x)\ \bar{s}^{N-1}_{\kappa,\lambda}(t). \tag{5.5}$$

By integrating the both sides of (5.5) over $[0,r]$ with respect to s, and then doing that over $[r,R]$ with respect to t, we conclude

$$\frac{\int_r^R \bar{\theta}_f(t,x)\,dt}{\int_0^r \bar{\theta}_f(s,x)\,ds} \leq \frac{s_{N,\kappa,\lambda}(R) - s_{N,\kappa,\lambda}(r)}{s_{N,\kappa,\lambda}(r)}.$$

From Lemma 5.1, we derive

$$\begin{aligned}\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} &= 1 + \frac{\int_{\partial M}\int_r^R \bar{\theta}_f(t,x)\,dt\,d\,\mathrm{vol}_h}{\int_{\partial M}\int_0^r \bar{\theta}_f(s,x)\,ds\,d\,\mathrm{vol}_h} \\ &\leq 1 + \frac{s_{N,\kappa,\lambda}(R) - s_{N,\kappa,\lambda}(r)}{s_{N,\kappa,\lambda}(r)} = \frac{s_{N,\kappa,\lambda}(R)}{s_{N,\kappa,\lambda}(r)}.\end{aligned}$$

This proves the theorem. □

In [40], Theorem 5.4 has been proved when $f = 0$ and $N = n$.

In the case of $N = \infty$, we have:

Theorem 5.5. *Let* M *be a connected complete Riemannian manifold with boundary, and let* $f : M \to \mathbb{R}$ *be a smooth function. Suppose that* ∂M *is compact. Suppose* $\mathrm{Ric}^\infty_{f,M} \geq 0$ *and* $H_{f,\partial M} \geq 0$. *Then for all* $r, R \in (0,\infty)$ *with* $r \leq R$, *we have*

$$\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} \leq \frac{R}{r}. \tag{5.6}$$

Proof. By Lemma 3.2, for all $s, t \in [0,\infty)$ with $s \leq t$, we have $\bar{\theta}_f(t,x) \leq \bar{\theta}_f(s,x)$. Integrating the both sides over $[0,r]$ with respect to s, and then doing that over $[r,R]$ with respect to t, we see

$$r \int_r^R \bar{\theta}_f(t,x)\,dt \leq (R-r) \int_0^r \bar{\theta}_f(s,x)\,ds.$$

By Lemma 5.1, we complete the proof. □

Remark 5.2. In [40], the author has proved a measure contraction inequality around the boundary when $f = 0$ and $N = n$. We can prove similar measure contraction inequalities in our setting. The measure contraction inequalities enable us to give another proof of Theorem 5.4, and of Theorem 5.5.

5.3. **Volume growth rigidity.** We have the following lemma:

Lemma 5.6. *Suppose that ∂M is compact. For $N \in [n,\infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Assume that there exists $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ such that for every $r \in (0,R]$ the equality in* (5.4) *holds. Then we have $\tau \geq R$ on ∂M.*

Proof. The proof is by contradiction. Suppose that a point $x_0 \in \partial M$ satisfies $\tau(x_0) < R$. Put $t_0 := \tau(x_0)$, and take $\epsilon > 0$ satisfying $t_0 + \epsilon < R$. By the continuity of τ, there exists a closed geodesic ball B in ∂M centered at x_0 such that for all $x \in B$ we have $\tau(x) \leq t_0 + \epsilon$. Lemma 3.1 implies that $m_f(B_R(\partial M))$ is not larger than

$$\int_{\partial M \setminus B} \int_0^{\min\{R,\tau(x)\}} s^{N-1}_{\kappa,\lambda}(t)\, dt\, d\, m_{f,\partial M} + \int_B \int_0^{t_0+\epsilon} s^{N-1}_{\kappa,\lambda}(t)\, dt\, d\, m_{f,\partial M}.$$

This is smaller than $m_{f,\partial M}(\partial M)\, s_{N,\kappa,\lambda}(R)$. On the other hand, $s_{N,\kappa,\lambda}(R)$ is equal to $m_f(B_R(\partial M))/m_{f,\partial M}(\partial M)$. This is a contradiction. □

In the case of $N = \infty$, we have:

Lemma 5.7. *Suppose that ∂M is compact. Suppose $\mathrm{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. Assume that there exists $R \in (0,\infty)$ such that for every $r \in (0,R]$ the equality in* (5.6) *holds. Then we have $\tau \geq R$ on ∂M.*

Proof. Suppose that for some $x_0 \in \partial M$ we have $\tau(x_0) < R$. Put $t_0 := \tau(x_0)$, and take $\epsilon > 0$ with $t_0 + \epsilon < R$. The continuity of τ implies that there exists a closed geodesic ball B in ∂M centered at x_0 such that τ is smaller than or equal to $t_0 + \epsilon$ on B. By Lemma 3.2,

$$m_f(B_R(\partial M)) \leq R\, m_{f,\partial M}(\partial M \setminus B) + (t_0+\epsilon)\, m_{f,\partial M}(B) < R\, m_{f,\partial M}(\partial M).$$

On the other hand, $m_f(B_R(\partial M))/m_{f,\partial M}(\partial M)$ is equal to R. This is a contradiction. We conclude the lemma. □

Suppose that ∂M is compact. Suppose that for $N \in [n,\infty)$ we have $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$ (resp. $\mathrm{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$), and that there exists $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ (resp. $R \in (0,\infty)$) such that for every $r \in (0,R]$ the equality in (5.4) (resp. (5.6)) holds. In this case, for every $r \in (0,R)$ the level set $\rho^{-1}_{\partial M}(r)$ is an $(n-1)$-dimensional submanifold of M (see Lemmas 5.6 and 5.7). In particular, $(B_r(\partial M), g)$ is an n-dimensional (not necessarily, connected) complete Riemannian manifold with boundary. We denote by $d_{B_r(\partial M)}$ and by $d_{\kappa,\lambda,r}$ the Riemannian distances on $(B_r(\partial M), g)$ and on $[0,r] \times_{\kappa,\lambda} \partial M$, respectively.

Lemma 5.8. *Suppose that ∂M is compact. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Assume that there exists $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ such that for every $r \in (0, R]$ the equality in* (5.4) *holds. Then for every $r \in (0, R)$, the metric space $(B_r(\partial M), d_{B_r(\partial M)})$ is isometric to $([0, r] \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda,r})$, and for every $x \in \partial M$ we have $f \circ \gamma_x = f(x) - \log s_{\kappa,\lambda}$ on $[0, r]$.*

Proof. Since each connected component of ∂M one-to-one corresponds to the connected component of $B_r(\partial M)$, it suffices to consider the case where $B_r(\partial M)$ is connected. For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all i we see $Y_{x,i} = s_{\kappa,\lambda}\, E_{x,i}$ on $[0, \min\{R, \bar{C}_{\kappa,\lambda}\}]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, $f \circ \gamma_x = f(x) - (N - n) \log s_{\kappa,\lambda}$ on $[0, \min\{R, \bar{C}_{\kappa,\lambda}\}]$ (see Remark 5.1). Define a map $\Phi : [0, r] \times \partial M \to B_r(\partial M)$ by $\Phi(t, x) := \gamma_x(t)$. For each $p \in (0, r) \times \partial M$ the map $D(\Phi|_{(0,r)\times\partial M})_p$ sends an orthonormal basis of $T_p([0, r] \times \partial M)$ to that of $T_{\Phi(p)}B_r(\partial M)$, and for each $x \in \{0, r\} \times \partial M$ the map $D(\Phi|_{\{0,r\}\times\partial M})_x$ sends an orthonormal basis of $T_x(\{0, r\} \times \partial M)$ to that of $T_{\Phi(x)}\partial(B_r(\partial M))$. Hence, Φ is a Riemannian isometry with boundary from $[0, r] \times_{\kappa,\lambda} \partial M$ to $B_r(\partial M)$. □

In the case of $N = \infty$, we have:

Lemma 5.9. *Suppose that ∂M is compact. Suppose $\mathrm{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. Assume that there exists $R \in (0, \infty)$ such that for every $r \in (0, R]$ the equality in* (5.6) *holds. Then for every $r \in (0, R)$ the metric space $(B_r(\partial M), d_{B_r(\partial M)})$ is isometric to $([0, r] \times \partial M, d_{[0,r]\times\partial M})$.*

Proof. We may assume that $B_r(\partial M)$ is connected. For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all i we have $Y_{x,i} = E_{x,i}$ on $[0, r]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$ (see Remark 5.1). Define a map $\Phi : [0, r] \times \partial M \to B_r(\partial M)$ by $\Phi(t, x) := \gamma_x(t)$. We see that Φ is a Riemannian isometry with boundary from $[0, r] \times \partial M$ to $B_r(\partial M)$. □

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that ∂M is compact. For $N \in [n, \infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N - 1)\kappa$ and $H_{f,\partial M} \geq (N - 1)\lambda$. Suppose (1.1).

By Lemma 5.2 and Theorem 5.4, for all $r, R \in (0,\infty)$ with $r \leq R$,

$$\frac{m_f(B_R(\partial M))}{s_{N,\kappa,\lambda}(R)} = \frac{m_f(B_r(\partial M))}{s_{N,\kappa,\lambda}(r)} = m_{f,\partial M}(\partial M).$$

If κ and λ satisfy the ball-condition, then for $R = C_{\kappa,\lambda}$, and for every $r \in (0, R]$ the equality in (5.4) holds; in particular, Lemmas 4.1 and 5.6 imply that τ is equal to $C_{\kappa,\lambda}$ on ∂M. If κ and λ do not satisfy the ball-condition, then for every $R \in (0,\infty)$, and for every $r \in (0, R]$ the equality in (5.4) holds; in particular, Lemma 5.6 implies that $\tau = \infty$ on ∂M. It follows that τ coincides with $\bar{C}_{\kappa,\lambda}$ on ∂M. From Lemma 5.8, for every $x \in \partial M$ we derive $f \circ \gamma_x = f(x) - (N-n) \log s_{\kappa,\lambda}$ on $I_{\kappa,\lambda}$.

If κ and λ satisfy the ball-condition, then $D(M, \partial M) = C_{\kappa,\lambda}$. By Lemma 2.3, M is compact. There exists $p \in M$ with $\rho_{\partial M}(p) = C_{\kappa,\lambda}$. Due to Theorem 1.1, (M, d_M) is isometric to $(B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})$ and $N = n$.

If κ and λ do not satisfy the ball-condition, then $\operatorname{Cut} \partial M = \emptyset$. It follows that ∂M is connected. Take a sequence $\{r_i\}$ with $r_i \to \infty$. By Lemma 5.8, for each r_i, there exists a Riemannian isometry $\Phi_i : [0, r_i] \times \partial M \to B_{r_i}(\partial M)$ with boundary from $[0, r_i] \times_{\kappa,\lambda} \partial M$ to $B_{r_i}(\partial M)$ defined by $\Phi_i(t, x) := \gamma_x(t)$. Since $\operatorname{Cut} \partial M = \emptyset$, we obtain a Riemannian isometry $\Phi : [0,\infty) \times \partial M \to M$ with boundary from $[0,\infty) \times_{\kappa,\lambda} \partial M$ to M defined by $\Phi(t, x) := \gamma_x(t)$ such that $\Phi|_{[0,r_i] \times \partial M} = \Phi_i$ for all r_i. This proves Theorem 1.2. □

Next, we prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that ∂M is compact. Suppose $\operatorname{Ric}^{\infty}_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. Furthermore, we assume (1.2).

By Lemma 5.3 and Theorem 5.5, for all $R \in (0,\infty)$ and $r \in (0, R]$,

$$\frac{m_f(B_R(\partial M))}{R} = \frac{m_f(B_r(\partial M))}{r} = m_{f,\partial M}(\partial M).$$

For every $R \in (0,\infty)$, and for every $r \in (0, R]$ the equality in (5.6) holds. From Lemma 5.7, it follows that $\tau = \infty$ on ∂M. We have $\operatorname{Cut} \partial M = \emptyset$, and hence ∂M is connected. Take a sequence $\{r_i\}$ with $r_i \to \infty$. Lemma 5.9 implies that for each r_i there exists a Riemannian isometry $\Phi_i : [0, r_i] \times \partial M \to B_{r_i}(\partial M)$ with boundary from $[0, r_i] \times \partial M$ to $B_{r_i}(\partial M)$ defined by $\Phi_i(t, x) := \gamma_x(t)$. Since $\operatorname{Cut} \partial M = \emptyset$, we obtain a Riemannian isometry $\Phi : [0,\infty) \times \partial M \to M$ with boundary from $[0,\infty) \times \partial M$ to M defined by $\Phi(t, x) := \gamma_x(t)$ such that $\Phi|_{[0,r_i] \times \partial M} = \Phi_i$ for all r_i. This proves Theorem 1.3. □

6. Splitting theorems

Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function.

6.1. **Main splitting theorems.** We prove Theorem 1.4.

Proof of Theorem 1.4. Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Suppose that for some $x_0 \in \partial M$ we have $\tau(x_0) = \infty$.

For the connected component ∂M_0 of ∂M containing x_0, we put

$$\Omega := \{y \in \partial M_0 \mid \tau(y) = \infty\}.$$

The assumption implies that Ω is non-empty. From the continuity of τ, it follows that Ω is closed in ∂M_0.

We show the openness of Ω in ∂M_0. Fix $y_0 \in \Omega$. Take $l \in (0, \infty)$, and put $p_0 := \gamma_{y_0}(l)$. There exists an open neighborhood U of p_0 in $\mathrm{Int}\, M$ contained in $D_{\partial M}$. Taking U smaller, we may assume that for each point $q \in U$ the unique foot point on ∂M of q belongs to ∂M_0. By Lemma 2.7, there exists $\epsilon \in (0, \infty)$ such that for all $q \in B_\epsilon(p_0)$, all asymptotes for γ_{y_0} from q lie in $\mathrm{Int}\, M$. We may assume $U \subset B_\epsilon(p_0)$. Fix $q_0 \in U$, and take an asymptote $\gamma_{q_0} : [0, \infty) \to M$ for γ_{y_0} from q_0. For $t \in (0, \infty)$, define a function $b_{\gamma_{y_0},t} : M \to \mathbb{R}$ by

$$b_{\gamma_{y_0},t}(p) := b_{\gamma_{y_0}}(q_0) + t - d_M(p, \gamma_{q_0}(t)).$$

We see that $b_{\gamma_{y_0},t} - \rho_{\partial M}$ is a support function of $b_{\gamma_{y_0}} - \rho_{\partial M}$ at q_0. Since γ_{q_0} lie in $\mathrm{Int}\, M$, for every $t \in (0, \infty)$ the function $b_{\gamma_{y_0},t}$ is smooth on a neighborhood of q_0. From Lemma 2.8, we deduce $\Delta_f b_{\gamma_{y_0},t}(q_0) \leq (N-1)(s'_\kappa(t)/s_\kappa(t))$. Note that $s'_\kappa(t)/s_\kappa(t) \to \lambda$ as $t \to \infty$. Furthermore, $\rho_{\partial M}$ is smooth on U, and by Lemma 3.3 we have $\Delta_f \rho_{\partial M} \geq (N-1)\lambda$ on U. Hence, $b_{\gamma_{y_0}} - \rho_{\partial M}$ is f-subharmonic on U. Since $b_{\gamma_{y_0}} - \rho_{\partial M}$ takes the maximal value 0 at p_0, Lemma 2.9 implies $b_{\gamma_{y_0}} = \rho_{\partial M}$ on U. From Lemma 2.6, it follows that Ω is open in ∂M_0.

Since ∂M_0 is a connected component of ∂M, we have $\Omega = \partial M_0$. By Lemma 2.4, ∂M is connected and $\mathrm{Cut}\, \partial M = \emptyset$. The equality in Lemma 3.3 holds on $\mathrm{Int}\, M$. For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all i we see $Y_{x,i} = s_{\kappa,\lambda} E_{x,i}$ on $[0, \infty)$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, for all $t \in [0, \infty)$ we have $(f \circ \gamma_x)(t) = f(x) + (N-n)\lambda t$ (see Remark 3.3). Define a map $\Phi : [0, \infty) \times \partial M \to M$ by $\Phi(t, x) := \gamma_x(t)$. For every $p \in (0, \infty) \times \partial M$ the map $D(\Phi|_{(0,\infty)\times \partial M})_p$ sends an orthonormal basis

of $T_p((0,\infty)\times\partial M)$ to that of $T_{\Phi(p)}M$, and for every $x\in\{0\}\times\partial M$ the map $D(\Phi|_{\{0\}\times\partial M})_x$ sends an orthonormal basis of $T_x(\{0\}\times\partial M)$ to that of $T_{\Phi(x)}\partial M$. Hence, Φ is a Riemannian isometry with boundary from $[0,\infty)\times_{\kappa,\lambda}\partial M$ to M. This proves Theorem 1.4. $\square$

Next, we prove Theorem 1.5.

Proof of Theorem 1.5. Assume $\sup f(M)<\infty$. Suppose $\mathrm{Ric}^{\infty}_{f,M}\geq 0$ and $H_{f,\partial M}\geq 0$. Let $x_0\in\partial M$ satisfy $\tau(x_0)=\infty$.

For the connected component ∂M_0 of ∂M containing x_0, we put

$$\Omega:=\{y\in\partial M_0\mid\tau(y)=\infty\}.$$

The assumption and the continuity of τ imply that Ω is a non-empty closed subset of ∂M_0.

We prove the openness of Ω in ∂M_0. Fix $y_0\in\Omega$. Take $l\in(0,\infty)$, and put $p_0:=\gamma_{y_0}(l)$. There exists an open neighborhood U of p_0 in $\mathrm{Int}\,M$ contained in $D_{\partial M}$. We may assume that for each point $q\in U$ the unique foot point on ∂M of q belongs to ∂M_0. By Lemma 2.7, there exists $\epsilon\in(0,\infty)$ such that for all $q\in B_\epsilon(p_0)$, all asymptotes for γ_{y_0} from q lie in $\mathrm{Int}\,M$. We may assume $U\subset B_\epsilon(p_0)$. By Lemma 2.10, $b_{\gamma_{y_0}}$ is f-subharmonic on U. Furthermore, $\rho_{\partial M}$ is smooth on U, and Lemma 3.4 implies $\Delta_f\rho_{\partial M}\geq 0$ on U. Therefore, $b_{\gamma_{y_0}}-\rho_{\partial M}$ is f-subharmonic on U. Since $b_{\gamma_{y_0}}-\rho_{\partial M}$ takes the maximal value 0 at p_0, Lemma 2.9 implies $b_{\gamma_{y_0}}=\rho_{\partial M}$ on U. By Lemma 2.6, Ω is open in ∂M_0.

Since ∂M_0 is a connected component of ∂M, we have $\Omega=\partial M_0$. By Lemma 2.4, ∂M is connected and $\mathrm{Cut}\,\partial M=\emptyset$. The equality in Lemma 3.4 holds on $\mathrm{Int}\,M$. For each $x\in\partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0)=e_{x,i}$ and $Y'_{x,i}(0)=-A_{u_x}e_{x,i}$. For all i we see $Y_{x,i}=E_{x,i}$ on $[0,\infty)$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0)=e_{x,i}$ (see Remark 3.3). Hence, the map $\Phi:[0,\infty)\times\partial M\to M$ defined by $\Phi(t,x):=\gamma_x(t)$ is a Riemannian isometry with boundary from $[0,\infty)\times\partial M$ to M. This completes the proof of Theorem 1.5. $\square$

Lemma 2.3 and the continuity of τ imply that if ∂M is compact and M is non-compact, then for some $x_0\in\partial M$ we have $\tau(x_0)=\infty$. By Theorems 1.4 and 1.5, we have the following rigidity results that have been proved in [20] (see also [11]) when $f=0$ and $N=n$.

Corollary 6.1. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f:M\to\mathbb{R}$ be a smooth function. Let $\kappa\leq 0$ and $\lambda:=\sqrt{|\kappa|}$. For $N\in[n,\infty)$, we suppose $\mathrm{Ric}^N_{f,M}\geq(N-1)\kappa$ and $H_{f,\partial M}\geq(N-1)\lambda$. If M is non-compact and*

∂M is compact, then (M, d_M) is isometric to $([0,\infty)\times_{\kappa,\lambda}\partial M, d_{\kappa,\lambda})$, and for all $x \in \partial M$ and $t \in [0,\infty)$ we have $(f \circ \gamma_x)(t) = f(x) + (N-n)\lambda t$.

Corollary 6.2. *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function such that $\sup f(M) < \infty$. Suppose $\mathrm{Ric}^{\infty}_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. If M is non-compact and ∂M is compact, then the metric space (M, d_M) is isometric to $([0,\infty)\times \partial M, d_{[0,\infty)\times\partial M})$.*

6.2. **Weighted Ricci curvature on the boundary.** Let h be the induced Riemannian metric on ∂M. For $x \in \partial M$, and for a unit vector u in $T_x\partial M$, we denote by $K_g(u_x, u)$ the sectional curvature at x in (M, g) determined by u_x and u.

It seems that the following is well-known, especially in a submanifold setting (see e.g., Proposition 9.36 in [4], and Lemma 5.4 in [40]).

Lemma 6.3. *Take $x \in \partial M$, and a unit vector u in $T_x\partial M$. Choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$ with $e_{x,1} = u$. Then we have*

$$\mathrm{Ric}_h(u) = \mathrm{Ric}_g(u) - K_g(u_x, u) + \operatorname{trace} A_{S(u,u)} - \sum_{i=1}^{n-1} \|S(u, e_{x,i})\|^2.$$

For all $x \in \partial M$ and $u \in T_x\partial M$, we see

$$h((\nabla(f|_{\partial M}))_x, u) = g((\nabla f)_x, u), \tag{6.1}$$

$$\mathrm{Hess}(f|_{\partial M})(u,u) = \mathrm{Hess}\, f(u,u) + g\left((\nabla f)_x, u_x\right)\, g\left(S(u,u), u_x\right). \tag{6.2}$$

We show the following:

Lemma 6.4. *Take $x \in \partial M$, and a unit vector u in $T_x\partial M$. Choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$ with $e_{x,1} = u$. Then for all $N \in [n, \infty)$, we have*

$$\begin{aligned}\mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) = {} & \mathrm{Ric}^{N}_{f}(u) + g((\nabla f)_x, u_x)\, g(S(u,u), u_x) \\ & - K_g(u_x, u) + \operatorname{trace} A_{S(u,u)} - \sum_{i=1}^{n-1} \|S(u, e_{x,i})\|^2.\end{aligned} \tag{6.3}$$

Proof. Assume $N \in (n, \infty)$. By (6.1) and (6.2), we have

$$\mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) = \mathrm{Ric}_h(u) + \mathrm{Hess}(f|_{\partial M})(u,u) - \frac{h((\nabla(f|_{\partial M}))_x, u)^2}{(N-1)-(n-1)}$$

$$= \mathrm{Ric}_h(u) + \mathrm{Hess}\, f(u,u) + g((\nabla f)_x, u_x)\, g(S(u,u), u_x) - \frac{g((\nabla f)_x, u)^2}{N-n}.$$

By Lemma 6.3, we see (6.3).

Assume $N = n$. If f is constant, then we see $\mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) = \mathrm{Ric}_h(u)$ and $\mathrm{Ric}^N_f(u) = \mathrm{Ric}_g(u)$, and hence Lemma 6.3 implies (6.3). If f is not constant, then both the left hand side of (6.3) and the right hand side are equal to $-\infty$. Therefore, we complete the proof. $\square$

In the case of $N = \infty$, we have:

Lemma 6.5. *Take $x \in \partial M$, and a unit vector u in $T_x \partial M$. Choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$ with $e_{x,1} = u$. Then we have*

$$\begin{aligned} \mathrm{Ric}^{\infty}_{f|_{\partial M}}(u) = \mathrm{Ric}^{\infty}_f(u) &+ g((\nabla f)_x, u_x)\, g(S(u,u), u_x) \\ &- K_g(u_x, u) + \operatorname{trace} A_{S(u,u)} - \sum_{i=1}^{n-1} \|S(u, e_{x,i})\|^2. \end{aligned} \tag{6.4}$$

Proof. From (6.2), it follows that

$$\begin{aligned} \mathrm{Ric}^{\infty}_{f|_{\partial M}}(u) &= \mathrm{Ric}_h(u) + \mathrm{Hess}(f|_{\partial M})(u,u) \\ &= \mathrm{Ric}_h(u) + \mathrm{Hess}\, f(u,u) + g((\nabla f)_x, u_x)\, g(S(u,u), u_x). \end{aligned}$$

Using Lemma 6.3, we have (6.4). $\square$

6.3. **Multi-splitting.** By Lemma 6.4, we see the following:

Lemma 6.6. *For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq 0$. If (M, d_M) is isometric to $([0,\infty) \times \partial M, d_{[0,\infty)\times \partial M})$, then $\mathrm{Ric}^{N-1}_{f|_{\partial M}, \partial M} \geq 0$.*

Proof. There exists a Riemannian isometry with boundary from M to $[0,\infty) \times \partial M$. Take $x \in \partial M$, and choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all i we have $Y_{x,i} = E_{x,i}$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. We see $A_{u_x} e_{x,i} = 0_x$ and $Y''_{x,1}(0) = 0_x$; in particular, $\operatorname{trace} A_{u_x} = 0$ and $K_g(u_x, e_{x,1}) = 0$. For all i,j we have $S(e_{x,i}, e_{x,j}) = 0_x$. By (6.3) and $\mathrm{Ric}^N_{f,M} \geq 0$, we have $\mathrm{Ric}^{N-1}_{f|_{\partial M}, \partial M} \geq 0$. $\square$

Let M_0 be a connected complete Riemannian manifold (without boundary). A normal geodesic $\gamma : \mathbb{R} \to M_0$ is said to be a *line* if for all $s, t \in \mathbb{R}$ we have $d_{M_0}(\gamma(s), \gamma(t)) = |s - t|$.

Fang, Li and Zhang [12] have proved the following splitting theorem of Cheeger-Gromoll type (see Theorem 1.3 in [12]):

Theorem 6.7 ([12])**.** *Let M_0 be an n-dimensional, connected complete Riemannian manifold, and let $f : M_0 \to \mathbb{R}$ be a smooth function. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M_0} \geq 0$. If M_0 contains a line, then there exists an $(n-1)$-dimensional Riemannian manifold N_0 such that M_0 is isometric to the standard product $\mathbb{R} \times N_0$.*

We have the following corollary of Theorem 1.4:

Corollary 6.8. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. If for some $x_0 \in \partial M$ we have $\tau(x_0) = \infty$, then there exist $k \in \{0, \dots, n-1\}$ and an $(n-1-k)$-dimensional, connected complete Riemannian manifold N_0 containing no line such that $(\partial M, d_{\partial M})$ is isometric to $(\mathbb{R}^k \times N_0, d_{\mathbb{R}^k \times N_0})$. In particular, (M, d_M) is isometric to $([0,\infty) \times \mathbb{R}^k \times N_0, d_{[0,\infty)\times\mathbb{R}^k\times N_0})$.*

Proof. Due to Theorem 1.4, the metric space (M, d_M) is isometric to $([0,\infty)\times\partial M, d_{[0,\infty)\times\partial M})$. Lemma 6.6 implies $\mathrm{Ric}^{N-1}_{f|_{\partial M},\partial M} \geq 0$. Applying Theorem 6.7 to ∂M inductively, we complete the proof. □

In the case of $N = \infty$, we see:

Lemma 6.9. *If $\mathrm{Ric}^\infty_{f,M} \geq 0$, and if the metric space (M, d_M) is isometric to $([0,\infty) \times \partial M, d_{[0,\infty)\times\partial M})$, then $\mathrm{Ric}^\infty_{f|_{\partial M},\partial M} \geq 0$.*

Proof. There exists a Riemannian isometry with boundary from M to $[0,\infty) \times \partial M$. Take $x \in \partial M$, and choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all i we have $Y_{x,i} = E_{x,i}$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. This implies $A_{u_x} e_{x,i} = 0_x$ and $Y''_{x,1}(0) = 0_x$. Hence, $\operatorname{trace} A_{u_x} = 0$ and $K_g(u_x, e_{x,1}) = 0$. For all i, j we see $S(e_{x,i}, e_{x,j}) = 0_x$. From (6.4), and from $\mathrm{Ric}^\infty_{f,M} \geq 0$, we deduce $\mathrm{Ric}^\infty_{f|_{\partial M},\partial M} \geq 0$. □

Fang, Li and Zhang [12] have proved the following splitting theorem of Cheeger-Gromoll type (see Theorem 1.1 in [12]):

Theorem 6.10 ([12])**.** *Let M_0 be an n-dimensional, connected complete Riemannian manifold, and let $f : M_0 \to \mathbb{R}$ be a smooth function such that $\sup f(M_0) < \infty$. Suppose $\mathrm{Ric}^\infty_{f,M_0} \geq 0$. If M_0 contains a line, then there exists an $(n-1)$-dimensional Riemannian manifold N_0 such that M_0 is isometric to the standard product $\mathbb{R} \times N_0$.*

Remark 6.1. Lichnerowicz [31] has proved Theorem 6.10 under the assumption that f is bounded.

In the case of $N = \infty$, we have the following:

Corollary 6.11. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function such that $\sup f(M) < \infty$. Suppose $\mathrm{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$.*

If for some $x_0 \in \partial M$ we have $\tau(x_0) = \infty$, then there exist $k \in \{0, \dots, n-1\}$ and an $(n-1-k)$-dimensional, connected complete Riemannian manifold N_0 containing no line such that $(\partial M, d_{\partial M})$ is isometric to $(\mathbb{R}^k \times N_0, d_{\mathbb{R}^k \times N_0})$. In particular, (M, d_M) is isometric to $([0,\infty) \times \mathbb{R}^k \times N_0, d_{[0,\infty)\times\mathbb{R}^k\times N_0})$.

Proof. By Theorem 1.5, (M, d_M) is isometric to $([0,\infty)\times\partial M, d_{[0,\infty)\times\partial M})$. From Lemma 6.9, we derive $\mathrm{Ric}^{\infty}_{f|_{\partial M},\partial M} \geq 0$. Notice that $\sup_{x\in\partial M} f(x)$ is finite. By using Theorem 6.10, we obtain the corollary. $\square$

6.4. **Variants of splitting theorems.** We have already known several rigidity results studied in [20] (and [11], [18]) for manifolds with boundary whose boundaries are disconnected. We study generalizations of the results in [20] (and [11], [18]).

The following has been proved in [20] (see Lemma 1.6 in [20]):

Lemma 6.12 ([20])**.** *Suppose that ∂M is disconnected. Let $\{\partial M_i\}_{i=1,2,\dots}$ denote the connected components of ∂M. Assume that ∂M_1 is compact. Put $D := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. Then there exists a connected component ∂M_2 of ∂M such that $d_M(\partial M_1, \partial M_2) = D$. Furthermore, for every $i = 1, 2$ there exists $x_i \in \partial M_i$ such that $d_M(x_1, x_2) = D$. The normal minimal geodesic $\gamma : [0, D] \to M$ from x_1 to x_2 is orthogonal to ∂M both at x_1 and at x_2, and the restriction $\gamma|_{(0,D)}$ lies in $\mathrm{Int}\, M$.*

First, we prove the following:

Theorem 6.13. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is disconnected. Let $\{\partial M_i\}_{i=1,2,\dots}$ denote the connected components of ∂M. Assume that ∂M_1 is compact. Put $D := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. For $N \in [n, \infty]$, we suppose $\mathrm{Ric}^N_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. Then (M, d_M) is isometric to $([0, D]\times\partial M_1, d_{[0,D]\times\partial M_1})$. Moreover, if $N \in [n, \infty)$, then for every $x \in \partial M_1$ the function $f \circ \gamma_x$ is constant on $[0, D]$.*

Proof. By Lemma 6.12, there exists a connected component ∂M_2 of ∂M such that $d_M(\partial M_1, \partial M_2) = D$. For each $i = 1, 2$, let $\rho_{\partial M_i} : M \to \mathbb{R}$ be the distance function from ∂M_i defined as $\rho_{\partial M_i}(p) := d_M(p, \partial M_i)$. Put

$$\Omega := \{p \in \mathrm{Int}\, M \mid \rho_{\partial M_1}(p) + \rho_{\partial M_2}(p) = D\}.$$

Lemma 6.12 implies that Ω is a non-empty closed subset of $\mathrm{Int}\, M$.

We show that Ω is open in $\mathrm{Int}\, M$. Take $p \in \Omega$. For each $i = 1, 2$, there exists a foot point $x_{p,i} \in \partial M_i$ on ∂M_i of p such that $d_M(p, x_{p,i}) = \rho_{\partial M_i}(p)$. From the triangle inequality, we derive $d_M(x_{p,1}, x_{p,2}) = D$. The normal minimal geodesic $\gamma : [0, D] \to M$ from $x_{p,1}$ to $x_{p,2}$ is

orthogonal to ∂M at $x_{p,1}$ and at $x_{p,2}$. Furthermore, $\gamma|_{(0,D)}$ lies in $\mathrm{Int}\, M$ and passes through p. There exists an open neighborhood U of p such that U is contained in $\mathrm{Int}\, M$ and $\rho_{\partial M_i}$ is smooth on U. By using Lemmas 3.1 and 3.2, we see $\Delta_f\, \rho_{\partial M_i} \geq 0$ on U; in particular, $-(\rho_{\partial M_1} + \rho_{\partial M_2})$ is f-subharmonic on U. By Lemma 2.9, Ω is open in $\mathrm{Int}\, M$.

Since $\mathrm{Int}\, M$ is connected, we have $\mathrm{Int}\, M = \Omega$. For each $x \in \partial M_1$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all i we see $Y_{x,i} = E_{x,i}$ on $[0, D]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, if $N \in [n, \infty)$, then $f \circ \gamma_x$ is constant on $[0, D]$ (see Remarks 3.1 and 3.2). We see that a map $\Phi : [0, D] \times \partial M_1 \to M$ defined by $\Phi(t, x) := \gamma_x(t)$ is a Riemannian isometry with boundary from $[0, D] \times \partial M_1$ to M. □

Next, we show the following:

Theorem 6.14. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is disconnected. Let $\{\partial M_i\}_{i=1,2,\dots}$ denote the connected components of ∂M. Assume that ∂M_1 is compact. Put $D := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. Let $\kappa > 0$. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then $\lambda < 0$ and $D \leq 2D_{\kappa,\lambda}$, where $D_{\kappa,\lambda} := \inf \{\, t > 0 \mid s'_{\kappa,\lambda}(t) = 0 \,\}$. If $D = 2D_{\kappa,\lambda}$, then (M, d_M) is isometric to $([0, D] \times_{\kappa,\lambda} \partial M_1, d_{[0,D] \times_{\kappa,\lambda} \partial M_1})$, and for every $x \in \partial M_1$ we have $f \circ \gamma_x = f(x) - (N-n) \log s_{\kappa,\lambda}$ on $[0, D]$.*

Proof. If $\lambda \geq 0$, then Theorem 6.13 implies that (M, d_M) is isometric to $([0, D] \times \partial M_1, d_{[0,D] \times \partial M_1})$, and for every $x \in \partial M_1$ the function $f \circ \gamma_x$ is constant on $[0, D]$. This contradicts the positivity of κ, and hence we have $\lambda < 0$.

We prove that if $D \geq 2D_{\kappa,\lambda}$, then the metric space (M, d_M) is isometric to $([0, 2D_{\kappa,\lambda}] \times_{\kappa,\lambda} \partial M_1, d_{[0,D_{\kappa,\lambda}] \times_{\kappa,\lambda} \partial M_1})$, and for every $x \in \partial M_1$ we have $f \circ \gamma_x = f(x) - (N-n) \log s_{\kappa,\lambda}$ on $[0, 2D_{\kappa,\lambda}]$. Assume $D \geq 2D_{\kappa,\lambda}$. By Lemma 6.12, there exists a connected component ∂M_2 of ∂M such that $d_M(\partial M_1, \partial M_2) = D$. For each $i = 1, 2$, let $\rho_{\partial M_i} : M \to \mathbb{R}$ be the distance function from ∂M_i defined as $\rho_{\partial M_i}(p) := d_M(p, \partial M_i)$. Put

$$\Omega := \{p \in \mathrm{Int}\, M \mid \rho_{\partial M_1}(p) + \rho_{\partial M_2}(p) = D\}.$$

The set Ω is a non-empty closed subset of $\mathrm{Int}\, M$.

We show that Ω is open in $\mathrm{Int}\, M$. Take $p \in \Omega$. For each $i = 1, 2$, we take a foot point $x_{p,i} \in \partial M_i$ on ∂M_i of p such that $d_M(p, x_{p,i}) = \rho_{\partial M_i}(p)$. From the triangle inequality, we derive $d_M(x_{p,1}, x_{p,2}) = D$. The normal

minimal geodesic $\gamma : [0, D] \to M$ from $x_{p,1}$ to $x_{p,2}$ is orthogonal to ∂M at $x_{p,1}$ and at $x_{p,2}$. Furthermore, $\gamma|_{(0,D)}$ lies in $\operatorname{Int} M$ and passes through p. There exists an open neighborhood U of p such that $\rho_{\partial M_i}$ is smooth on U. By using Lemma 3.1, for all $q \in U$, we see

$$
\begin{aligned}
(6.5)\quad -\frac{\Delta_f\ (\rho_{\partial M_1} + \rho_{\partial M_2})\,(q)}{N-1} &\le \frac{s'_{\kappa,\lambda}(\rho_{\partial M_1}(q))}{s_{\kappa,\lambda}(\rho_{\partial M_1}(q))} + \frac{s'_{\kappa,\lambda}(\rho_{\partial M_2}(q))}{s_{\kappa,\lambda}(\rho_{\partial M_2}(q))} \\
&= \frac{s'_{\kappa,\lambda}(\rho_{\partial M_1}(q) + \rho_{\partial M_2}(q)) - \lambda s_{\kappa,\lambda}(\rho_{\partial M_1}(q) + \rho_{\partial M_2}(q))}{s_{\kappa,\lambda}(\rho_{\partial M_1}(q))s_{\kappa,\lambda}(\rho_{\partial M_2}(q))}.
\end{aligned}
$$

Since $\kappa > 0$, the function $s'_{\kappa,\lambda}/s_{\kappa,\lambda}$ is monotone decreasing on $(0, C_{\kappa,\lambda})$, and satisfies $s'_{\kappa,\lambda}(2D_{\kappa,\lambda})/s_{\kappa,\lambda}(2D_{\kappa,\lambda}) = \lambda$. By the triangle inequality and the assumption $D \ge 2D_{\kappa,\lambda}$, we have $\rho_{\partial M_1} + \rho_{\partial M_2} \ge 2D_{\kappa,\lambda}$ on U. Therefore, by (6.5), $-(\rho_{\partial M_1} + \rho_{\partial M_2})$ is f-subharmonic on U. By Lemma 2.9, Ω is open in $\operatorname{Int} M$.

The connectedness of $\operatorname{Int} M$ implies $\operatorname{Int} M = \Omega$. For each $x \in \partial M_1$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all i we have $Y_{x,i} = s_{\kappa,\lambda} E_{x,i}$ on $[0, D]$, where $\{Y_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, $f \circ \gamma_x = f(x) - (N - n) \log s_{\kappa,\lambda}$ on $[0, D]$ (see Remark 3.1). We see $D = 2D_{\kappa,\lambda}$. A map $\Phi : [0, 2D_{\kappa,\lambda}] \times \partial M_1 \to M$ defined by $\Phi(t, x) := \gamma_x(t)$ is a Riemannian isometry with boundary from $[0, 2D_{\kappa,\lambda}] \times_{\kappa,\lambda} \partial M_1$ to M. $\square$

7. Eigenvalue rigidity

Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function.

7.1. **Lower bounds.**

We prove the inequalities (1.8) in Theorem 1.6 and (1.9) in Theorem 1.7.

Allegretto and Huang [1] have shown the following inequality of Picone type in a Euclidean setting (see Theorem 1.1 in [1]):

Lemma 7.1. *Let ϕ and ψ be functions on M that are smooth on a domain U in M, and satisfy $\phi > 0$ and $\psi \ge 0$ on U. Then for all $p \in (1, \infty)$ we have the following inequality on U:*

$$
(7.1)\qquad \|\nabla \psi\|^p \ge \|\nabla \phi\|^{p-2} g\left(\nabla\left(\psi^p\, \phi^{1-p}\right), \nabla \phi\right).
$$

Proof. For a fixed $p \in (1,\infty)$, we put $q := p(p-1)^{-1}$. By the Young inequality, we have

$$\|\nabla \psi\| \left(\frac{\psi \|\nabla \phi\|}{\phi}\right)^{p-1} \leq \frac{\|\nabla \psi\|^p}{p} + \frac{1}{q}\left(\frac{\psi \|\nabla \phi\|}{\phi}\right)^{p} \tag{7.2}$$

on U. By (7.2), and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\nabla \psi\|^p &\geq p\left(\psi \phi^{-1}\right)^{p-1} \|\nabla \psi\| \|\nabla \phi\|^{p-1} - (p-1)\left(\psi\phi^{-1}\right)^p \|\nabla \phi\|^p \\ &\geq p\left(\psi \phi^{-1}\right)^{p-1} g(\nabla \phi, \nabla \psi)\|\nabla \phi\|^{p-2} - (p-1)\left(\psi\phi^{-1}\right)^p \|\nabla \phi\|^p \\ &= \|\nabla \phi\|^{p-2} g\left(\nabla\left(\psi^p \phi^{1-p}\right), \nabla \phi\right). \end{aligned} \tag{7.3}$$

This completes the proof. □

Remark 7.1. In Lemma 7.1, we assume that the equality in (7.1) holds on U. In this case, the equalities in (7.3) also hold on U. From the equality in the Young inequality, and from that in the Cauchy-Schwarz inequality, we deduce that for some constant $c \neq 0$ we have $\phi\|\nabla\psi\| = \psi\|\nabla\phi\|$ and $\nabla\psi = c\nabla\phi$ on U; in particular, $\psi = c\,\phi$ on U.

Now, we prove the inequality (1.8) in Theorem 1.6.

Proposition 7.2. *Suppose that M is compact. Let $p \in (1,\infty)$. For $N \in [n,\infty)$, we suppose $\operatorname{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, assume $D(M,\partial M) \leq D$. Then we have* (1.8).

Proof. Let $\phi_{p,N,\kappa,\lambda,D} : [0,D] \to \mathbb{R}$ be a function satisfying (1.6) for $\mu = \mu_{p,N,\kappa,\lambda,D}$. We may assume $\phi_{p,N,\kappa,\lambda,D}|_{(0,D]} > 0$. The equation (1.6) is written in the form

$$\left(|\phi'(t)|^{p-2}\phi'(t) s^{N-1}_{\kappa,\lambda}(t)\right)' + \mu\, |\phi(t)|^{p-2}\phi(t) s^{N-1}_{\kappa,\lambda}(t) = 0,$$
$$\phi(0) = 0, \quad \phi'(D) = 0.$$

Therefore, it follows that $\phi'_{p,N,\kappa,\lambda,D}|_{[0,D)} > 0$. Put $\Phi := \phi_{p,N,\kappa,\lambda,D} \circ \rho_{\partial M}$. Take a non-negative, non-zero smooth function ψ on M whose support is compact and contained in $\operatorname{Int} M$. By Lemma 7.1, we have

$$\|\nabla \psi\|^p \geq \|\nabla \Phi\|^{p-2} g\left(\nabla\left(\psi^p\, \Phi^{1-p}\right), \nabla \Phi\right) \tag{7.4}$$

on $\operatorname{Int} M \setminus \operatorname{Cut} \partial M$. By using (7.4) and Proposition 3.7, we have

$$\begin{aligned} &\int_M \|\nabla \psi\|^p \, d\, m_f \geq \int_M \|\nabla \Phi\|^{p-2} g\left(\nabla\left(\psi^p\, \Phi^{1-p}\right), \nabla \Phi\right)\, d\, m_f \\ &\geq \int_M \left(\psi^p\, \Phi^{1-p}\right)\left(\left(-\left(\left(\phi'\right)^{p-1}\right)' - (N-1)\frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}\left(\phi'\right)^{p-1}\right)\circ \rho_{\partial M}\right)\, d\, m_f \\ &= \mu_{p,N,\kappa,\lambda,D} \int_M \psi^p \, d\, m_f. \end{aligned}$$

We obtain $R_{f,p}(\psi) \geq \mu_{p,N,\kappa,\lambda,D}$. This implies (1.8). □

Next, we prove the inequality (1.9) in Theorem 1.7.

Proposition 7.3. *Suppose that M is compact. Let $p \in (1,\infty)$. Suppose $\mathrm{Ric}^{\infty}_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. For $D \in (0,\infty)$, assume $D(M,\partial M) \leq D$. Then we have* (1.9).

Proof. Let $\phi_{p,\infty,D} : [0,D] \to \mathbb{R}$ be a function satisfying (1.7) for $\mu = \mu_{p,\infty,D}$. We may assume $\phi_{p,\infty,D}|_{(0,D]} > 0$. In this case, we have $\phi'_{p,\infty,D}|_{[0,D)} > 0$. Put $\Phi := \phi_{p,\infty,D} \circ \rho_{\partial M}$. Take a non-negative, non-zero smooth function ψ on M whose support is compact and contained in $\mathrm{Int}\, M$. By Lemma 7.1, we have

$$\|\nabla \psi\|^p \geq \|\nabla \Phi\|^{p-2} g\left(\nabla\left(\psi^p\, \Phi^{1-p}\right), \nabla \Phi\right) \tag{7.5}$$

on $\mathrm{Int}\, M \setminus \mathrm{Cut}\, \partial M$. By using (7.5) and Proposition 3.8, we have

$$\begin{aligned}
\int_M \|\nabla \psi\|^p \, d\, m_f &\geq \int_M \|\nabla \Phi\|^{p-2} g\left(\nabla\left(\psi^p\, \Phi^{1-p}\right), \nabla \Phi\right)\, d\, m_f \\
&\geq \int_M \left(\psi^p\, \Phi^{1-p}\right) \left(-\left((\phi')^{p-1}\right)' \circ \rho_{\partial M}\right) d\, m_f = \mu_{p,\infty,D} \int_M \psi^p \, d\, m_f.
\end{aligned}$$

We obtain $R_{f,p}(\psi) \geq \mu_{p,\infty,D}$. This implies (1.9). □

Remark 7.2. In Proposition 7.2 (resp. 7.3), we assume that there exists a non-negative, non-zero smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in $\mathrm{Int}\, M$ such that $R_{f,p}(\psi) = \mu_{p,N,\kappa,\lambda,D}$ (resp. $R_{f,p}(\psi) = \mu_{p,\infty,D}$). In this case, the equality in (7.4) (resp. (7.5)) holds on $\mathrm{Int}\, M \setminus \mathrm{Cut}\, \partial M$. Therefore, for some constant $c \neq 0$ we have $\psi = c\, \Phi$ on M (see Remark 7.1). Furthermore, the equality case in (3.11) (resp. 3.12) happens (see Remark 3.5).

7.2. **Equality cases.** We prove Theorems 1.6 and 1.7.

In the proofs, we use the following fact:

Proposition 7.4. *Suppose that M is compact. Let $p \in (1,\infty)$. Then there exists a non-negative, non-zero function Ψ in $W^{1,p}_0(M, m_f)$ such that $R_{f,p}(\Psi) = \mu_{f,1,p}(M)$. Moreover, for some $\alpha \in (0,1)$ the function Ψ is $C^{1,\alpha}$-Hölder continuous on M.*

Proposition 7.4 is well-known in the standard case where $f = 0$. In the standard case, the existence follows from the standard compactness argument, and the regularity follows from the results by Tolksdorf in [44]. The method of the proof also works in our weighted setting.

For $D \in (0,\infty)$, we put $S_D(\partial M) := \{\, q \in M \mid \rho_{\partial M}(q) = D \,\}$.

Kasue has shown the following in the proof of Theorem 2.1 in [21]:

Proposition 7.5 ([21]). *Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Suppose that M is compact. Assume that for some $D \in (0, \bar{C}_{\kappa,\lambda})$ we have $\operatorname{Cut}\partial M = S_D(\partial M)$. For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. Assume further that for all $x \in \partial M$ and i we have $Y_{x,i} = s_{\kappa,\lambda}\, E_{x,i}$ on $[0, D]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Then κ and λ satisfy the model-condition, (M, d_M) is a (κ, λ)-equational model space, and $D = D_{\kappa,\lambda}(M)$.*

Now, we prove Theorem 1.6.

Proof of Theorem 1.6. Suppose that M is compact. Let $p \in (1, \infty)$. For $N \in [n, \infty)$, we suppose $\operatorname{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, assume $D(M, \partial M) \leq D$. By Proposition 7.2, we have (1.8).

Assume that the equality in (1.8) holds. By Proposition 7.4, there exists a non-negative, non-zero function Ψ in $W_0^{1,p}(M, m_f)$ such that $R_{f,p}(\Psi) = \mu_{p,N,\kappa,\lambda,D}$ and Ψ is $C^{1,\alpha}$-Hölder continuous on M. Put $\Phi := \phi_{p,N,\kappa,\lambda,D} \circ \rho_{\partial M}$. Then Φ coincides with a constant multiplication of Ψ on M (see Remark 7.2); in particular, Φ is also $C^{1,\alpha}$-Hölder continuous.

For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all i we see $Y_{x,i} = s_{\kappa,\lambda}\, E_{x,i}$ on $[0, \tau(x)]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, $f \circ \gamma_x = f(x) - (N-n) \log s_{\kappa,\lambda}$ on $[0, \tau(x)]$ (see Remarks 3.5 and 7.2).

Let $D = \bar{C}_{\kappa,\lambda}$. Since D is finite, κ and λ satisfy the ball-condition and $D = C_{\kappa,\lambda}$. There exists $p_0 \in M$ such that $\rho_{\partial M}(p_0) = D(M, \partial M)$. Note that p_0 belongs to $\operatorname{Cut}\partial M$. Now, we prove $\rho_{\partial M}(p_0) = C_{\kappa,\lambda}$. We assume $\rho_{\partial M}(p_0) < C_{\kappa,\lambda}$. Let x_0 be a foot point on ∂M of p_0. From the property of Jacobi fields, p_0 is not the first conjugate point of ∂M along γ_{x_0}. Hence, $\rho_{\partial M}$ is not differentiable at p_0. Since Φ is $C^{1,\alpha}$-Hölder continuous, we see $\phi'_{p,N,\kappa,\lambda,D}(\rho_{\partial M}(p_0)) = 0$. From $\phi'_{p,N,\kappa,\lambda,D}|_{[0,D)} > 0$, we deduce $\rho_{\partial M}(p_0) = D$. This contradicts $D = C_{\kappa,\lambda}$. Therefore, $\rho_{\partial M}(p_0) = C_{\kappa,\lambda}$. By Theorem 1.1, (M, d_M) is isometric to $(B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})$ and $N = n$.

Let $D \in (0, \bar{C}_{\kappa,\lambda})$. We prove $\operatorname{Cut}\partial M = S_D(\partial M)$. Since $D(M, \partial M) \leq D$, we see $S_D(\partial M) \subset \operatorname{Cut}\partial M$. We show the opposite. Take $p_0 \in \operatorname{Cut}\partial M$. By the property of Jacobi fields, $\rho_{\partial M}$ is not differentiable at p_0. The regularity of Φ implies $\phi'_{p,N,\kappa,\lambda,D}(\rho_{\partial M}(p_0)) = 0$; in particular, $\rho_{\partial M}(p_0) = D$. We have $\operatorname{Cut}\partial M = S_D(\partial M)$. By Proposition 7.5, κ and λ satisfy the model-condition, (M, d_M) is a (κ, λ)-equational model

space, and $D = D_{\kappa,\lambda}(M)$. From $\tau = D_{\kappa,\lambda}(M)$ on ∂M, it follows that $f \circ \gamma_x = f(x) - (N-n)\log s_{\kappa,\lambda}$ on $[0, D_{\kappa,\lambda}(M)]$ for all $x \in \partial M$. We complete the proof of Theorem 1.6. $\square$

Remark 7.3. In [21], the proof of Theorem 1.6 in the standard case where $f = 0, N = n$ and $p = 2$ relies on the approximation theorem obtained by Greene and Wu in [16]. It seems that the approximation theorem in [16] does not work in our non-linear case of $p \neq 2$.

Next, we prove Theorem 1.7.

Proof of Theorem 1.7. Suppose that M is compact. Let $p \in (1,\infty)$. Suppose $\mathrm{Ric}^{\infty}_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. For $D \in (0,\infty)$, we assume $D(M,\partial M) \leq D$. By Proposition 7.3, we have (1.9).

Assume that the equality in (1.9) holds. By Proposition 7.4, there exists a non-negative, non-zero function Ψ in $W^{1,p}_0(M, m_f)$ such that $R_{f,p}(\Psi) = \mu_{p,\infty,D}$ and Ψ is $C^{1,\alpha}$-Hölder continuous on M. Put $\Phi := \phi_{p,\infty,D} \circ \rho_{\partial M}$. Then Φ coincides with a constant multiplication of Ψ on M (see Remark 7.2); in particular, Φ is also $C^{1,\alpha}$-Hölder continuous.

For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all i we have $Y_{x,i} = E_{x,i}$ on $[0, \tau(x)]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$ (see Remarks 3.5 and 7.2).

We prove $\mathrm{Cut}\,\partial M = S_D(\partial M)$. Since $D(M,\partial M) \leq D$, it holds that $S_D(\partial M) \subset \mathrm{Cut}\,\partial M$. We show the opposite. Take $p_0 \in \mathrm{Cut}\,\partial M$. By the property of Jacobi fields, $\rho_{\partial M}$ is not differentiable at p_0. By the regularity of Φ, we see $\phi'_{p,\infty,D}(\rho_{\partial M}(p_0)) = 0$; in particular, $\rho_{\partial M}(p_0) = D$. It follows that $\mathrm{Cut}\,\partial M = S_D(\partial M)$; in particular, $D(M,\partial M) = D$. By Proposition 7.5, we complete the proof of Theorem 1.7. $\square$

7.3. **Explicit lower bounds.** For $N \in [2,\infty)$ and $D \in (0,\infty)$, we see $\mu_{2,N,0,0,D} = \mu_{2,\infty,D} = \pi^2(2D)^{-2}$.

By Theorems 1.6 and 1.7, we have the following:

Corollary 7.6. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. For $N \in [n,\infty]$, we suppose $\mathrm{Ric}^N_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. For $D \in (0,\infty)$, we assume $D(M,\partial M) \leq D$. Then we have*

$$\mu_{f,1,2}(M) \geq \frac{\pi^2}{4D^2}. \tag{7.6}$$

If the equality in (7.6) *holds, then* $D(M,\partial M)=D$*, and* (M,d_M) *is a* $(0,0)$*-equational model space. Moreover, if* $N\in[n,\infty)$*, then for every* $x\in\partial M$ *the function* $f\circ\gamma_x$ *is constant on* $[0,D]$.

Li and Yau [30] have obtained (7.6) when $f=0$ and $N=n$.
Kasue [21] has proved the following (see Lemma 1.3 in [21]):

Lemma 7.7 ([21])**.** *For all* $N\in[2,\infty)$, $\kappa,\lambda\in\mathbb{R}$ *and* $D\in(0,\bar{C}_{\kappa,\lambda}]\setminus\{\infty\}$*, we have*

$$\mu_{2,N,\kappa,\lambda,D}>\left(4\max_{t\in[0,D]}\int_t^D s_{\kappa,\lambda}^{N-1}(s)\,ds\int_0^t s_{\kappa,\lambda}^{1-N}(s)\,ds\right)^{-1}.$$

In the case of $p=2$, by Theorem 1.6 and Lemma 7.7 we have:

Corollary 7.8. *Let* M *be an* n*-dimensional, connected complete Riemannian manifold with boundary, and let* $f:M\to\mathbb{R}$ *be a smooth function. Suppose that* M *is compact. For* $N\in[n,\infty)$*, we suppose* $\mathrm{Ric}_{f,M}^N\geq(N-1)\kappa$ *and* $H_{f,\partial M}\geq(N-1)\lambda$*. For* $D\in(0,\bar{C}_{\kappa,\lambda}]\setminus\{\infty\}$*, we assume* $D(M,\partial M)\leq D$*. Then we have*

$$\mu_{f,1,2}(M)>\left(4\max_{t\in[0,D]}\int_t^D s_{\kappa,\lambda}^{N-1}(s)\,ds\int_0^t s_{\kappa,\lambda}^{1-N}(s)\,ds\right)^{-1}.$$

8. First eigenvalue estimates

Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f:M\to\mathbb{R}$ be a smooth function.

8.1. **Area estimates.** Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M satisfying $\partial\Omega\cap\partial M=\emptyset$. For the canonical Riemannian volume measure $\mathrm{vol}_{\partial\Omega}$ on $\partial\Omega$, let $m_{f,\partial\Omega}:=e^{-f|_{\partial\Omega}}\,\mathrm{vol}_{\partial\Omega}$. Put

$$\delta_1(\Omega):=\inf_{p\in\Omega}\rho_{\partial M}(p),\quad \delta_2(\Omega):=\sup_{p\in\Omega}\rho_{\partial M}(p). \tag{8.1}$$

Kasue [22] has proved the following when $f=0$ and $N=n$.

Proposition 8.1. *For* $N\in[n,\infty)$*, we suppose* $\mathrm{Ric}_{f,M}^N\geq(N-1)\kappa$ *and* $H_{f,\partial M}\geq(N-1)\lambda$*. Let* Ω *be a relatively compact domain in* M *such that* $\partial\Omega$ *is a smooth hypersurface in* M *satisfying* $\partial\Omega\cap\partial M=\emptyset$*. Then*

$$m_f(\Omega)\leq m_{f,\partial\Omega}(\partial\Omega)\sup_{t\in(\delta_1(\Omega),\delta_2(\Omega))}\frac{\int_t^{\delta_2(\Omega)}s_{\kappa,\lambda}^{N-1}(s)\,ds}{s_{\kappa,\lambda}^{N-1}(t)}, \tag{8.2}$$

where $\delta_1(\Omega)$ *and* $\delta_2(\Omega)$ *are the values defined as* (8.1).

Proof. Define a function $\phi : [\delta_1(\Omega), \delta_2(\Omega)] \to \mathbb{R}$ by

$$\phi(t) := \int_{\delta_1(\Omega)}^{t} \frac{\int_s^{\delta_2(\Omega)} s_{\kappa,\lambda}^{N-1}(u)\,du}{s_{\kappa,\lambda}^{N-1}(s)}\,ds,$$

and put $\Phi := \phi \circ \rho_{\partial M}$. By Lemma 3.5, on $\mathrm{Int}\, M \setminus \mathrm{Cut}\, \partial M$

$$\Delta_{f,2}\, \Phi \geq 1. \tag{8.3}$$

By Lemma 2.5, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of compact subsets of $\bar{\Omega}$ satisfying that for every k, the set $\partial\Omega_k$ is a smooth hypersurface in M except for a null set in $(\partial\Omega, m_{f,\partial\Omega})$, and satisfying the following: (1) for all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $\bar{\Omega} \setminus \mathrm{Cut}\, \partial M = \bigcup_{k\in\mathbb{N}} \Omega_k$: (3) for every $k \in \mathbb{N}$, and for almost every point $p \in \partial\Omega_k \cap \partial\Omega$ in $(\partial\Omega, m_{f,\partial\Omega})$, there exists the unit outer normal vector for Ω_k at p that coincides with the unit outer normal vector on $\partial\Omega$ for Ω at p; (4) for every $k \in \mathbb{N}$, on $\partial\Omega_k \setminus \partial\Omega$, there exists the unit outer normal vector field ν_k for Ω_k such that $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.

For the canonical Riemannian volume measure vol_k on $\partial\Omega_k \setminus \partial\Omega$, put $m_{f,k} := e^{-f|_{\partial\Omega_k \setminus \partial\Omega}}\, \mathrm{vol}_k$. Let $\nu_{\partial\Omega}$ be the unit outer normal vector on $\partial\Omega$ for Ω. By integrating the both sides of (8.3) on Ω_k, and by the Green formula, we have

$$\begin{aligned} m_f\,(\Omega_k) &\leq \int_{\Omega_k} \Delta_{f,2}\, \Phi\, d\, m_f \\ &= -\int_{\partial\Omega_k \setminus \partial\Omega} g(\nu_k, \nabla\Phi)\, d\, m_{f,k} - \int_{\partial\Omega_k \cap \partial\Omega} g(\nu_{\partial\Omega}, \nabla\Phi)\, d\, m_{f,\partial\Omega}. \end{aligned}$$

Since $g(\nu_k, \nabla\Phi) \geq 0$ on $\partial\Omega_k \setminus \partial\Omega$, we have

$$m_f\,(\Omega_k) \leq -\int_{\partial\Omega_k \cap \partial\Omega} g(\nu_{\partial\Omega}, \nabla\Phi)\, d\, m_{f,\partial\Omega}.$$

Therefore, from the Cauchy-Schwarz inequality, we derive

$$\begin{aligned} m_f\,(\Omega_k) &\leq \int_{\partial\Omega_k \cap \partial\Omega} (\phi' \circ \rho_{\partial M})\, |g(\nu_{\partial\Omega}, \nabla\rho_{\partial M})|\, d\, m_{f,\partial\Omega} \\ &\leq m_{f,\partial\Omega}\,(\partial\Omega) \sup_{t\in(\delta_1(\Omega),\delta_2(\Omega))} \phi'(t). \end{aligned}$$

By letting $k \to \infty$, we have (8.2). □

Remark 8.1. In [22], the key points of the proof of Proposition 8.1 in the standard case where $f = 0$ and $N = n$ are to use the comparison theorem concerning a generalized Laplacian of $\rho_{\partial M}$ proved in [19], and to apply the approximation theorem in [16] to $\rho_{\partial M}$. We see that similar

theorems also hold in our weighted case. From this point of view, Proposition 8.1 can be proved in the same way as that in [22].

In the case of $N=\infty$, we have the following:

Proposition 8.2. *Suppose* $\operatorname{Ric}^{\infty}_{f,M}\geq 0$ *and* $H_{f,\partial M}\geq 0$. *Let* Ω *be a relatively compact domain in* M *such that* $\partial\Omega$ *is a smooth hypersurface in* M *satisfying* $\partial\Omega\cap\partial M=\emptyset$. *Then*

$$m_f(\Omega)\leq m_{f,\partial\Omega}\left(\partial\Omega\right)\,\left(\delta_2(\Omega)-\delta_1(\Omega)\right), \tag{8.4}$$

where $\delta_1(\Omega)$ *and* $\delta_2(\Omega)$ *are the values defined as* (8.1).

Proof. Define a function $\phi:[\delta_1(\Omega),\delta_2(\Omega)]\to\mathbb{R}$ by

$$\phi(t):=-\frac{t^2}{2}+\delta_2(\Omega)t-\delta_1(\Omega)\delta_2(\Omega)+\frac{\delta_1(\Omega)^2}{2},$$

and put $\Phi:=\phi\circ\rho_{\partial M}$. By Lemma 3.6, on $\operatorname{Int}M\setminus\operatorname{Cut}\partial M$

$$\Delta_{f,2}\,\Phi\geq 1. \tag{8.5}$$

By Lemma 2.5, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of compact subsets of $\bar{\Omega}$ satisfying that for every k, the set $\partial\Omega_k$ is a smooth hypersurface in M except for a null set in $(\partial\Omega,m_{f,\partial\Omega})$, satisfying the following: (1) for all $k_1,k_2\in\mathbb{N}$ with $k_1<k_2$, we have $\Omega_{k_1}\subset\Omega_{k_2}$; (2) $\bar{\Omega}\setminus\operatorname{Cut}\partial M=\bigcup_{k\in\mathbb{N}}\Omega_k$; (3) for every $k\in\mathbb{N}$, and for almost every point $p\in\partial\Omega_k\cap\partial\Omega$ in $(\partial\Omega,m_{f,\partial\Omega})$, there exists the unit outer normal vector for Ω_k at p that coincides with the unit outer normal vector on $\partial\Omega$ for Ω at p; (4) for every $k\in\mathbb{N}$, on $\partial\Omega_k\setminus\partial\Omega$, there exists the unit outer normal vector field ν_k for Ω_k such that $g(\nu_k,\nabla\rho_{\partial M})\geq 0$.

For the canonical Riemannian volume measure vol_k on $\partial\Omega_k\setminus\partial\Omega$, put $m_{f,k}:=e^{-f|_{\partial\Omega_k\setminus\partial\Omega}}\operatorname{vol}_k$. Let $\nu_{\partial\Omega}$ be the unit outer normal vector on $\partial\Omega$ for Ω. By integrating the both sides of (8.5) on Ω_k, and by the Green formula, we have

$$\begin{aligned} m_f\left(\Omega_k\right) &\leq \int_{\Omega_k}\Delta_{f,2}\,\Phi\,d\,m_f\\ &=-\int_{\partial\Omega_k\setminus\partial\Omega}g(\nu_k,\nabla\Phi)\,d\,m_{f,k}-\int_{\partial\Omega_k\cap\partial\Omega}g(\nu_{\partial\Omega},\nabla\Phi)\,d\,m_{f,\partial\Omega}.\end{aligned}$$

Since $g(\nu_k,\nabla\Phi)\geq 0$ on $\partial\Omega_k\setminus\partial\Omega$, we have

$$m_f\left(\Omega_k\right)\leq-\int_{\partial\Omega_k\cap\partial\Omega}g(\nu_{\partial\Omega},\nabla\Phi)\,d\,m_{f,\partial\Omega}.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} m_f(\Omega_k) &\le \int_{\partial\Omega_k \cap \partial\Omega} (\delta_2(\Omega) - \rho_{\partial M})\, |g(\nu_{\partial\Omega}, \nabla\rho_{\partial M})|\, d\, m_{f,\partial\Omega} \\ &\le m_{f,\partial\Omega}(\partial\Omega)\ (\delta_2(\Omega) - \delta_1(\Omega)). \end{aligned}$$

Letting $k \to \infty$, we obtain (8.4). □

8.2. **Eigenvalue estimates.** Let $\alpha \in (0,\infty)$. The *f-Dirichlet α-isoperimetric constant* $ID_\alpha(M, m_f)$ of M is defined as

$$ID_\alpha(M, m_f) := \inf_{\Omega} \frac{m_{f,\partial\Omega}(\partial\Omega)}{(m_f(\Omega))^{1/\alpha}},$$

where the infimum is taken over all relatively compact domains Ω in M such that $\partial\Omega$ are smooth hypersurfaces in M satisfying $\partial\Omega \cap \partial M = \emptyset$. The *$f$-Dirichlet α-Sobolev constant* $SD_\alpha(M, m_f)$ of M is defined as

$$SD_\alpha(M, m_f) := \inf_{\phi \in W_0^{1,1}(M,m_f)\setminus\{0\}} \frac{\int_M \|\nabla\phi\|\, d\, m_f}{\left(\int_M |\phi|^\alpha\, d\, m_f\right)^{1/\alpha}},$$

where the infimum is taken over all non-zero functions ϕ in $W_0^{1,1}(M, m_f)$.

The following relationship between the isoperimetric constant and the Sobolev constant has been formally established by Federer and Fleming in [14] (see e.g., [7], [29]), and later used by Cheeger in [8] for the estimate of the first Dirichlet eigenvalue of the Laplacian.

Proposition 8.3 ([14])**.** *For all $\alpha \in (0,\infty)$ we have*

$$ID_\alpha(M, m_f) = SD_\alpha(M, m_f).$$

A proof of Proposition 8.3 has been given in [29] in the case of $f = 0$ (see Theorem 9.5 in [29]). The method of the proof also works in our weighted setting.

For $N \in [2,\infty)$, $\kappa, \lambda \in \mathbb{R}$, and $D \in (0, \bar{C}_{\kappa,\lambda}]$, let $C(N,\kappa,\lambda,D)$ be a positive constant defined by

$$C(N,\kappa,\lambda,D) := \sup_{t\in[0,D)} \frac{\int_t^D s_{\kappa,\lambda}^{N-1}(s)\, ds}{s_{\kappa,\lambda}^{N-1}(t)}. \tag{8.6}$$

Notice that $C(N,\kappa,\lambda,\infty)$ is finite if and only if $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$; in this case, we have $C(N,\kappa,\lambda,D) = ((N-1)\lambda)^{-1}\left(1 - e^{-(N-1)\lambda D}\right)$; in particular, $(2\, C(N,\kappa,\lambda,\infty))^{-2} = ((N-1)\lambda/2)^2$.

By using Proposition 8.1, we obtain the following:

Theorem 8.4. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $p \in (1, \infty)$. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. For $D \in (0, \bar{C}_{\kappa,\lambda}]$, we assume $D(M, \partial M) \leq D$. Then we have*

$$\mu_{f,1,p}(M) \geq (\, p\, C(N, \kappa, \lambda, D)\,)^{-p}, \tag{8.7}$$

where $C(N, \kappa, \lambda, D)$ is the constant defined as (8.6).

Proof. Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M satisfying $\partial\Omega \cap \partial M = \emptyset$. By Proposition 8.1, we have

$$m_f(\Omega) \leq m_{f,\partial\Omega}(\partial\Omega)\, C(N, \kappa, \lambda, D).$$

By Proposition 8.3, we have $ID_1(M, m_f) = SD_1(M, m_f)$. We obtain $SD_1(M, m_f) \geq C(N, \kappa, \lambda, D)^{-1}$. Therefore, for all $\phi \in W^{1,1}_0(M, m_f)$

$$\int_M |\phi|\, d\, m_f \leq C(N, \kappa, \lambda, D) \int_M \|\nabla \phi\|\, d\, m_f. \tag{8.8}$$

Let ψ be a non-zero function in $W^{1,p}_0(M, m_f)$. Put $q := p\,(1-p)^{-1}$. In (8.8), by replacing ϕ with $|\psi|^p$, and by the Hölder inequality, we see

$$\begin{aligned}\int_M |\psi|^p\, d\, m_f &\leq p\, C(N, \kappa, \lambda, D) \int_M |\psi|^{p-1}\, \|\nabla \psi\|\, d\, m_f \\ &\leq p\, C(N, \kappa, \lambda, D) \left(\int_M |\psi|^p\, d\, m_f \right)^{1/q} \left(\int_M \|\nabla \psi\|^p\, d\, m_f \right)^{1/p}.\end{aligned}$$

Considering the Rayleigh quotient $R_{f,p}(\psi)$, we obtain (8.7). $\square$

In the case of $N = \infty$, we have the following:

Theorem 8.5. *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $p \in (1, \infty)$. Suppose $\mathrm{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. For $D \in (0, \infty]$, we assume $D(M, \partial M) \leq D$. Then we have*

$$\mu_{f,1,p}(M) \geq (\, pD\,)^{-p}. \tag{8.9}$$

Proof. Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M satisfying $\partial\Omega \cap \partial M = \emptyset$. Proposition 8.2 implies $m_f(\Omega) \leq m_{f,\partial\Omega}(\partial\Omega)\, D$. From Proposition 8.3, we derive $SD_1(M, m_f) \geq D^{-1}$. Therefore, for all $\phi \in W^{1,1}_0(M, m_f)$

$$\int_M |\phi|\, d\, m_f \leq D \int_M \|\nabla \phi\|\, d\, m_f. \tag{8.10}$$

Take a non-zero function ψ in $W_0^{1,p}(M, m_f)$. Put $q := p\,(1-p)^{-1}$. In (8.10), by replacing ϕ with $|\psi|^p$, and by the Hölder inequality, we see

$$\int_M |\psi|^p \, d\,m_f \leq p\,D \left(\int_M |\psi|^p \, d\,m_f \right)^{1/q} \left(\int_M \|\nabla \psi\|^p \, d\,m_f \right)^{1/p}.$$

Considering the Rayleigh quotient $R_{f,p}(\psi)$, we obtain (8.9). □

Now, we prove Theorem 1.8.

Proof of Theorem 1.8. Suppose that ∂M is compact. Let $p \in (1, \infty)$. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in [n, \infty)$, we suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. We have

$$C(N, \kappa, \lambda, D) = ((N-1)\lambda)^{-1} \left(1 - e^{-(N-1)\lambda\, D}\right).$$

The right hand side is monotone increasing as $D \to \infty$. From Theorem 8.4, we derive (1.10).

Assume that the equality in (1.10) holds. By Theorem 8.4, we have $D = \infty$. Since ∂M is compact, M is non-compact. By Corollary 6.1, (M, d_M) is isometric to $([0, \infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$, and for all $x \in \partial M$ and $t \in [0, \infty)$ we have $(f \circ \gamma_x)(t) = f(x) + (N-n)\lambda t$. This completes the proof of Theorem 1.8. □

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